WUHAN UNIVERSITY

Lecture Notes

General Relativity

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Prerequisite:

Familiarity with classical mechanics and special relativity.

Field theory and differential geometry background are not must, but they will be very beneficial for the course.

If you have any question, please send an email to me before the course starts.

Textbooks:

Robert M. Wald, General relativity, University of Chicago press, 1984.

Sean. M. Carroll, Spacetime and Geometry, Addison-Wesley, 2003.

These books can be borrowed from the library or bought from online retailers.

References:

David Kay, Schaums Outline of Tensor Calculus, McGraw-Hill, 2011.

Eric Poisson, An advanced course in general relativity

Two Chinese books:

Liu Liao, General relativity, Higher Education Press, ISBN: 9787040144307

Liang Canbin, Zhou Bin, Differential geometry and general relativity, Science Press, ISBN: (Vol.1) 9787030164605; (Vol.2) 9787030240576; (Vol.3) 9787030252319

Contents

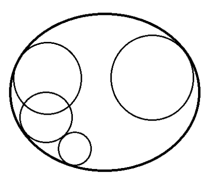
Chapter 1

Manifold

One of the most fundamental mathematical object in this course. All our spacetimes studied in this course are manifold.
• Open ball in $\mathbb{R}^n$ ($\mathbb{R}^n$ is a topological space)
An open ball with radius r centered around a point $y = (y^1, \ldots, y^n)$ in $\mathbb{R}^n$ is

$$\left\{ x \in \mathbb{R}^n \,\middle|\, |x - y| = \Big(\sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \Big)^{\frac{1}{2}} < r \right\}$$

• Open set in $\mathbb{R}^n$: set of points expressable as union of open balls.



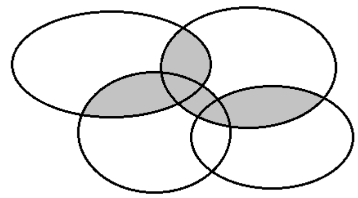
1.1 manifold

Definition 1.1.1 (Mainfold)**.** *A set made up of pieces "look like" open subsets of $\mathbb{R}^n$ such that these pieces can be "sewn/glued together" smoothly.*

Definition 1.1.2 (Precise Version)**.** *An n-dimensional, C^∞, real manifold $\mathcal{M}$ is a set together with a collection of subsets $\{O_\alpha\}$ satisfying:*
(1)Each $p \in \mathcal{M}$ must lie in at least one $\{O_\alpha\}$. $\{O_\alpha\}$ cover $\mathcal{M}$.
(2)For each α, there is one-to-one, onto, map:

$$\psi_\alpha : O_\alpha \to U_\alpha \tag{1.1}$$

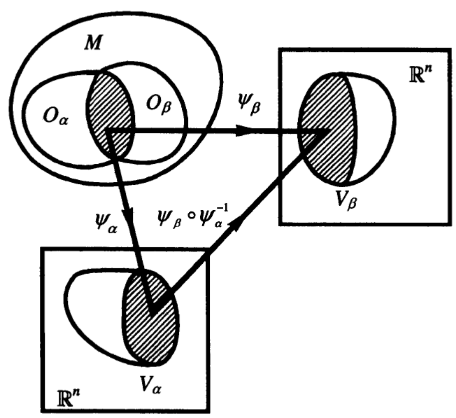
where U_α is an open subset of $\mathbb{R}^n$.
(3)For two sets $O_\alpha \bigcap O_\beta \neq \emptyset$, the map $\psi_\beta \circ \psi_\alpha^{-1}$ is C^∞.



Now let us look at this definition more closely.
What kind of set?
• $\mathcal{M}$ is a set: a collection of objects.
• Together there should exist O_α. Each $O_\alpha \subseteq \mathcal{M}$.
• O_α covers $\mathcal{M}$: Any object in $\mathcal{M}$ must be in one or more O_α.
What kind of subset?
• One-to-one and onto mappable to open subset of $\mathbb{R}^n$("one-to-one" & "one-to-one correspondance" are different).
• one-to-one + onto = one-to-one correspondance;
• one-to-one: Let $f : A \to B$ be a function. f is said to be one-to-one if

$$\forall x_1, x_2 \in A, \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

• onto: $\forall y \in B, \exists x \in A, s.t, f(x) = y$.
• Composite of $\psi_\beta \circ \psi_\alpha^{-1}$ is smooth (infinity differentiable).



1.2 Example of manifolds

(1)

$$\mathbb{R}^n, \{O\} \tag{1.2}$$
$$O = \mathbb{R}^n, \psi = \text{identity map} \tag{1.3}$$

(2) any open subset of $\mathbb{R}^n$.
(3) 2-sphere: S^2

$$S^2 = \left\{(x^1, x^2, x^3) \in \mathbb{R} \,\middle|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\right\} \tag{1.4}$$

One function ψ can not map S^2 to $\mathbb{R}^2$. And this is also the reason we need a collection of ψ_α.
(4) In this course, we view the spacetime as a 4-d manifold. There are theories with extra-dimensions, such as string theory. They view the spacetime as a 10- or 11- dimensional manifold.

this course $\Rightarrow$ spacetime is 4-d manifold.

Now the definition of manifold is clear. Let us also name other quantities we encountered in this definition.
• The mapping ψ_α are called "chart" or "coordinate system".
• The collection of all (O_α, ψ_α) is called "atlas".

1.3 Product of manifold

Given two manifold $\mathcal{M}, \mathcal{M}'$ with dimensions d, d', form a new manifold.

For

$$\begin{aligned} p \in O_\alpha \subseteq \mathcal{M}, p' \in O_\beta \subseteq \mathcal{M} \\ \psi_\alpha : O_\alpha \to U_\alpha, \psi'_\beta : O'_\beta \to U'_\beta \end{aligned}$$

then the new manifold is $\mathcal{M} \times \mathcal{M}'$, s.t.,

$$\begin{aligned} (p, p') \in O_\alpha \times O'_\beta \subseteq \mathcal{M} \times \mathcal{M}' \\ \psi_{\alpha\beta} : O_\alpha \times O'_\beta \to U_\alpha \times U'_\beta \\ \psi_{\alpha\beta}(p, p') = [\psi_\alpha(p), \psi'_\beta(p)] \end{aligned}$$

This $\psi_{\alpha\beta}$ and $\{O_{\alpha\beta}\}$ satisfies the definition of a manifold. Then $\mathcal{M} \times \mathcal{M}'$ is a new manifold. It is called the "product manifold"

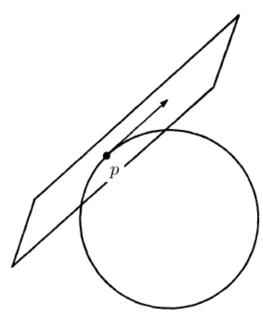
• Most manifold in this course are $\mathbb{R}^n \times S^m, n + m = 4$.
• Many manifold in string theory are "product manifold" too.

Chapter 2

Vectors

2.1 Tangent vectors and tangent space

• Intuitively, when manifold is embedded in $\mathbb{R}^n$.
• When embedding is not explicit or possible.



Let $\mathcal{F} = \{$ all C^∞ functions from $\mathcal{M} \to \mathbb{R}^1\}$

Then a tangent vector v at point $p \in \mathcal{M}$ is a map $v : \mathcal{F} \to \mathbb{R}^1$, s.t.

$$v(af + bg) = av(f) + bv(g) \quad \forall f, g \in \mathcal{F}; a, b \in \mathbb{R} \tag{2.1}$$
$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{(Leibnitz rule)} \tag{2.2}$$

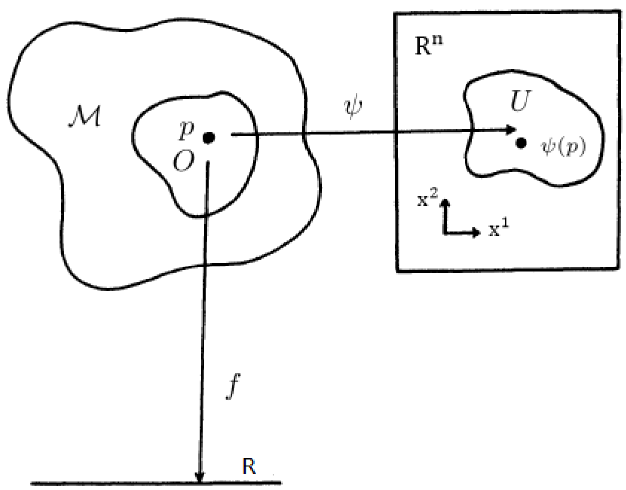
(**Homework:** Show that $v(c) = 0$ for constant function C.)
The collection of tangent vectors form a vector space, V_p.

Theorem 2.1.1. *Let $\mathcal{M}$ be a n-dimensional manifold. Let $p \in \mathcal{M}$ and V_p denote the tangent vector space at p, then $dimV_p = n$.*

The operators $X_\mu : \mathcal{F} \to \mathbb{R}$ defined as

$$X_\mu \circ f \equiv \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}), \quad \mu = 1, \ldots, n \tag{2.3}$$

span the tangent space V_p and therefore is its basis.
Here $(x^1, x^2, \ldots, x^n)$ are the Cartesian coordinates of $\mathbb{R}^n$. ψ is the chart from $\mathcal{M}$ to $\mathbb{R}^n$.



• $\left\{X^\mu = \dfrac{\partial}{\partial x^\mu}, \mu = 1, \ldots, n\right\}$ is called the "coordinate basis" (Because it has to apply on $f \circ \psi^{-1}$, which means ψ^{-1} has to be used and known), and ψ is called "coordinate system".

Proof. Let $\psi : O \to U \subset \mathbb{R}^n$ be a chart, $p \in O$. Let $f \in \mathcal{F}$, then

$$f \circ \psi^{-1} : U \to \mathbb{R}^1 \textit{ is } C^\infty \tag{2.4}$$

Then for $\mu = 1, \ldots, n$ define $X_\mu : \mathcal{F} \to \mathbb{R}^1$ by

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)} \tag{2.5}$$

We see X_μ $(\mu = 1, \ldots, n)$ are tangent vectors from definition Eq(2.1) and Eq(2.2), and they are linearly independent.
We show that X_μ span V_p.
For any C^∞ function, $\mathcal{F} : \mathbb{R}^n \to R$, we can construct $H_\mu(x)$, s.t.

$$F(x) = F(a) + \sum_{\mu=1}^{n}(x^\mu - a^\mu)H_\mu(x) \quad x = (x^1, x^2, \ldots, x^n), a = (a^1, a^2, \ldots, a^n) \tag{2.6}$$

Then we have

$$H_\mu(a) = \frac{\partial F}{\partial x^\mu}\Big|_{x=a} \tag{2.7}$$

Now letting $F = f \circ \psi^{-1}$, $a = \psi(p)$, then for all $q \in O$, putting F on $\psi(q) \equiv x$ yields

$$f(q) = f(p) + \sum_{\mu=1}^{n} [x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] H_\mu\Big(\psi(q)\Big) \tag{2.8}$$

here $x^\mu \circ \psi(\cdot)$ is the coordinate projection operator.

Let $v \in V_p$, we show v is a linear combination of x_n. To do this, we apply v on f, and then evaluate on p

$$v[f(q)]\Big|_{q=p} = v[f(p)] + \sum_{\mu=1}^{n} \Big\{ \Big[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)\Big] v(H \circ \psi) + H_\mu \circ \psi(q) v\Big[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)\Big]\Big\}\Big|_{q=p}$$

Now v acting on constant produces zero, therefore,

$$v[f(q)] = \sum_{\mu=1}^{n} [H_\mu \circ \psi(p)] v(x^\mu \circ \psi) \tag{2.9}$$

Now, $H_\mu \circ \psi(p) = H_\mu(a) = \frac{\partial F}{\partial x^\mu}\Big|_{x=\mu} = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)} = X_\mu(f)$

then $v(f) = \sum_{\mu=1}^{n} v^\mu X_\mu(f)$, where $v^\mu \equiv v(x^\mu \circ \psi)$.

This can be done for any v, and therefore an arbitrary vector v can be expressed as a sum of X_μ,

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu \tag{2.10}$$

□

Comments:

1. X_μ depend on both ψ and f.
2. v^μ only depend on ψ.
3. Therefore v depend on f too, but its component does not depend on f, they only depend on ψ.
4. Note that v is indeed a number in $\mathbb{R}^1$.

If we choose another coordinate system ψ'

$$\psi' \to X'_\nu$$

then chain rule applies,

$$X_\mu = \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu}\Big|_{\psi(p)} X'_\nu \tag{2.11}$$

where x'^ν is the ν-th component of $\psi' \circ \psi^{-1}$.
For a vector v, the components change according to

$$v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \quad \text{(vector transformation law)}$$

(So they are the same as change of basis in linear algebra, where $\dfrac{\partial x'^\nu}{\partial x^\mu}$ is the matrix for coordinate change.)

2.2 Smooth curve & tangent field

A smooth curve on a manifold $\mathcal{M}$ is a C^∞ map

$$C : \mathbb{R} \to \mathcal{M}$$

For any point $p \in \mathcal{M}$ lying on the curve C, and $\forall f \in \mathcal{F}$, we associate a tangent vector $\mathcal{T}_p$, defined as

$$\mathcal{T}_p(f) \equiv \frac{d(f \circ C)}{dt} = \sum_\mu \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \frac{dx^\mu}{dt} = \sum_\mu \frac{dx^\mu}{dt} X_\mu(f). \tag{2.12}$$

Note here $f \circ C$ is a function from $\mathbb{R}^1$ to $\mathbb{R}^1$; $f \circ \psi^{-1}$ maps from $\mathbb{R}^n$ to $\mathbb{R}^1$. This tangent vector function apparently depend on the choice of f.

However, its components in any coordinate basis, is independent of f and given by

$$T = \sum_\mu T^\mu X_\mu(f); \quad T^\mu = \frac{dx^\mu}{dt} \tag{2.13}$$

2.3 Tangent field

• V_p, V_q are different vector space.
• If no other structure is given to $\mathcal{M}$, then no natural way to identity $v_p \in V_p$ with $v_q \in V_q$.
• However, we can define a smooth tangent field on all points on $\mathcal{M}$.
(1) A tangent field, v, on a manifold $\mathcal{M}$ is an assignment of a tangent vector, $v|_p \in V_p$ at each point $p \in \mathcal{M}$.
(2) Now for $f \in \mathcal{F} : \mathcal{M} \to \mathbb{R}$, for each $p \in \mathcal{M}$, $v|_p(f)$ is a number; i.e., $v(f)$ is a function on $\mathcal{M}$.
(3) If $v(f)$ is a smooth function for any f, then we say the tangent field v is smooth.
(4) v is smooth only when v^μ is smooth because

$$v = v^\mu X_\mu(f) \; . \tag{2.14}$$

and $X_\mu(f)$ is smooth because f, ψ^{-1} are smooth.

Chapter 3

Tensor

3.1 Dual vector space

Definition 3.1.1 (Dual vector space)**.** *Let V be a finite-dimensional vector space (Which could be a tangent space), let V^* be collection of linear maps $f : V \to \mathbb{R}^1$.*
Define addition and scalar multiplication in this space, and then V^* becomes a linear space. We name it as ***dual vector space*** *to V. Any $v^* \in V$ is called a* ***dual vector****.*

If $v_1, \ldots, v_n$ is a basis of V, then $v^{1*}, \ldots, v^{2*}$, s.t.,

$$v^{\mu *}(v_\nu) = \delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

can be proven to be ***basis of*** *V^**

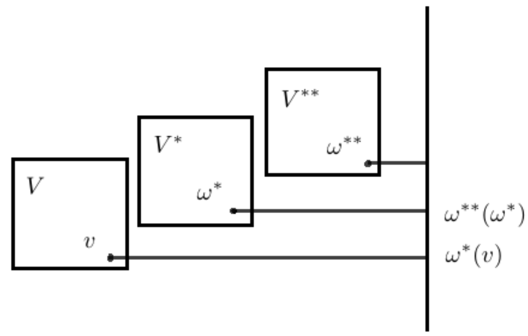
Definition 3.1.2 (Double dual vector space)**.**

$$V^{**} = \{f | f : v^* \to \mathbb{R}^1\}$$

Now we do the follow identification

$$\begin{gathered} \omega^{**}(\omega^*) = \omega^*(v) \\ \omega^{**} \longleftrightarrow v \end{gathered}$$

Then the double dual vector space can be identified with the space V.



3.2 Tensor

Definition 3.2.1 (Tensor). *A tensor of type (k,l) over V is*

$$T : \underbrace{V \times V \times \cdots \times V}_{k} \times \underbrace{V^* \times \cdots \times V^*}_{l} \longrightarrow \mathbb{R}^1$$

• *$\mathcal{T} = \{T\}$is an n^{k+l} dimensional vector space;*
• *each element T is multilinear;*
• *any T can be completely known if we know how it acts on the basis $\{v^*\}$ of V^* and $\{v\}$ of V.*

$$T \longleftarrow T(v^{*i_1}, v^{*i_2}, \dots, v^{*i_k}; v_{j_1}, \dots, v_{j_l})$$

Two tensor operations

Definition 3.2.2 (Contraction).

$$C : \mathcal{T}(k,l) \longrightarrow \mathcal{T}(k-1, l-1)$$

$$CT = \sum_{\sigma=1}^{n} T(\dots, v^{\sigma *}, \dots; \dots, v_\sigma, \dots)$$

Change the vectors at ith, jth positions to a pair of basis.
Contraction between ith dual and jth positions.
• *Contraction is independent of change of basis for V, V^*.*

Definition 3.2.3 (Outer product).

$$T \otimes T' \equiv T \cdot T'$$

$$\mathcal{T}(k+k', l+l') \equiv \left\{ T \otimes T' \,\middle|\, T \in \mathcal{T}(k,l); T' \in \mathcal{T}(k',l') \right\}$$

• *Outer product form a new tensor. Their collection forms a vector space too.*
• $\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes \cdots \otimes v^{\nu_1^*} \otimes \cdots \otimes v^{v_l^*}\}$ *forms a basis for* $\mathcal{T}(k,l)$.
Every T *of type* (k,l) *can be expressed as*

$$T = \sum_{\mu_1 \ldots \nu_l = 1}^{n} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} v_{\mu_1} \otimes \cdots \otimes v^{\nu_l^*}$$

$T^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_l}$ *are called* ***components*** *of* T.

Definition 3.2.4 (Contraction and out product in terms of components)**.** *Contraction in terms of components:*

$$(CT)^{\mu_1 \cdots \mu_{k-1}}{}_{\nu_1 \ldots \nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1 \ldots \sigma \ldots \mu_k}{}_{\nu_1 \ldots \sigma \ldots \nu_{k-1}}$$

Out product in terms of components:

$$S = T \otimes T'$$
$$S^{\mu_1 \cdots \mu_{k+k'}}{}_{\nu_1 \ldots \nu_{l+l'}} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l} T'^{\mu_k+1 \ldots \mu_{k+k'}}{}_{\nu_{l+1} \ldots \nu_{l+l'}}$$

we will mostly work with components in this course.

Applying to tangent space V_p of $p \in \mathcal{M}$
• V_p^* is called cotangent space;
• $v \in V_p^*$ is called contravariant vectors;
$v \in V_p^*$ is called covariant vectors;
convention: contravariant vector component use superscript v^μ;
convariant vector component use subscript v_μ
• V_p basis:

$$\partial/\partial x', \partial/\partial x^2, \ldots, \partial/\partial x^n$$

V_p^* basis denoted as $dx^1, \ldots, dx^n$(only symbols; defined through $dx^\mu(\dfrac{\partial}{\partial x^\nu}) = \delta^\mu_\nu$).
• Change of basis

$$v'^{\mu'} = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^{\mu'}}{\partial x^\mu} \tag{3.1}$$

$$\omega'_{\mu'} = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^{\mu'}} \tag{3.2}$$

$$T'^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \sum_{\mu_1,\ldots,\nu_l=1}^{n} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \cdot \cdots \cdot \frac{\partial x^{\nu_l}}{\partial x'^{\nu'_l}} \tag{3.3}$$

• Tensor field on $\mathcal{M} = \{$one tensor from each point on $\mathcal{M}\}$. A tensor field is called "smooth" if $T(\omega^1, \ldots, \omega^k; v_1, \ldots, v_l)$ is smooth, where $\omega^1, \ldots, \omega^k$ are arbitrary k smooth covariant vector field, $v_1, \ldots, v_l$ are arbitrary l smooth covariant vector field.

Metric tensor

A metric tensor, g, on a manifold $\mathcal{M}$, is a symmetric, nondegenerate tensor of type $(0,2)$.

• symmetric

$$g(v_1, v_2) = g(v_2, v_1) \quad v_i \in V_p$$

• non-degenerate:

$$\text{if } g(v, v_1) = 0 \text{ for all } v \in V_p, \text{ then } v_1 = 0 \quad .$$

So a metric is a inner product on the tangent space at each point.

Notation:

$$g = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu \otimes dx^\nu$$

Sometimes, we write g as ds^2, omitting $\otimes$, so that

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$$

• $g_{\mu\nu}$ is non-degenerate, therefore it is diagonalizable to

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

This is called **signature** of the metric. Positive definite metric is called **Riemannian**. Diag$(-+++)$ is called **Lorentzian** (Spacetime has this signature).

• g as a map and its inverse

$$g_{\mu\nu} : V_p \times V_p \longrightarrow \mathbb{R}^1 \qquad \text{type (0,2) tensor}$$

For a $v \in V_p$ fixed, then $g(\cdot, v) : V_p \longrightarrow \mathbb{R}^1$; therefore,

$$g(\cdot, v) \in V^*$$

In another words, $g : V \longrightarrow V^*$, $\sum_\mu g_{\mu\nu} v^\mu$ is a dual vector for any vector v^μ.

This map is one-to-one correspondant. Then an inverse map exist

$$g^{-1} : V^* \longrightarrow V \qquad g^{-1} \text{ is type (2,0)}$$

Components are written as $g^{\mu\nu}$. By definition, we also require

$$\sum_\nu g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma \tag{3.4}$$

Since $\sum_\mu g_{\mu\nu} v^\mu$ is a vector in dual space, we use v_ν to denote its components. $v_\nu = \sum_\mu g_{\mu\nu} v^\mu$.
Similarly,

$$g : T(k,l) \longrightarrow T(k-1, l+1)$$

In component notation,

$$\sum_\sigma g_{\sigma\eta} T^{\mu_1 \ldots \mu_i \sigma \mu_{i+2} \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = T^{\mu_1 \ldots \mu_i}{}_\eta{}^{\mu_{i+2}, \ldots \mu_l}{}_{\nu_1 \ldots \nu_l}$$

Similarly, $g^{-1} : T(k,l) \longrightarrow T(k+1, l-1)$.

And in component notation,

$$\sum_\sigma g^{\sigma\eta} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_i \sigma \nu_{i+2} \ldots \nu_l} = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_i}{}^\eta{}_{\nu_{i+2} \ldots \nu_l}$$

Notation:
Symmetric tensor built from any tensor of type $(0,l)$

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$$

$$T_{(\mu_1 \ldots \mu_l)} = \frac{1}{l!} \sum_{\pi \in perm} T_{\mu_{\pi(1)} \ldots \mu_{\pi(l)}}$$

perm here is the permutation of $\{1, \ldots, l\}$.
Anti-symmetric tensor built from $(0,l)$ type

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$$

$$T_{[\mu_1 \ldots \mu_l]} = \frac{1}{l!} \sum_{\pi \in perm} \delta_\pi T_{\mu_{\pi(1)} \ldots \mu_{\pi(l)}}$$

where $\delta_\pi = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$.

We can build partially symmetric, anti-symmetric tensors, e.g.,

$$T^{(\mu\nu)\sigma}{}_{[\alpha\beta]} = \frac{1}{4}[T^{\mu\nu\sigma}{}_{\alpha\beta} + T^{\nu\mu\sigma}{}_{\alpha\beta} - T^{\mu\nu\sigma}{}_{\beta\alpha} - T^{\nu\mu\sigma}{}_{\beta\alpha}]$$

A totally anti-symmetric tensor

$$T_{\mu_1 \ldots \mu_l} = T_{[\mu_1 \ldots \mu_l]}$$

is called a differential l-form.

Einstein summation rule

For any contraction, or summation over particular indices, we omit the sum symbol $\sum_{n_1,\ldots,n_l}$. So when two indices repeatly appear in a term, they are summed.
E.g.,

$$T^{\mu\nu}S_{\mu\sigma}{}^{\eta} \equiv \sum_{\mu} T^{\mu\nu}S_{\mu\sigma}{}^{\eta}$$

Chapter 4

Curvature

4.1 Derivative

4.1.1 We seek a covariant derivative operator, ∇.

(1) ∇: T of type $(k, l) \longrightarrow T$ of type $(k, l+1)$.
(2) Linearity

$$\begin{aligned} &\nabla_\mu(aT^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} + bS^{\sigma_1 \ldots \sigma_m}{}_{\eta_1 \ldots \eta_n}) \\ &= a\nabla_\mu T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} + b\nabla_\mu S^{\sigma_1 \ldots \sigma_m}{}_{\eta_1 \ldots \eta_n} \end{aligned} \tag{4.1}$$

where $a, b \in \mathbb{R}^1$.
(3) Leibnitz rule

$$\nabla_\mu(T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} S^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}) = (\nabla_\mu T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l}) S^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} + T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} \nabla_\mu S^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$$

(4) Commutativity with contraction

$$\nabla_\mu(T^{\alpha_1 \ldots \nu \ldots \alpha_l}{}_{\beta_1 \ldots \nu \ldots \beta_k}) = \nabla_\mu T^{\alpha_1 \ldots \nu \ldots \alpha_l}{}_{\beta_1 \ldots \nu \ldots \beta_k} \tag{4.2}$$

(5) For scalar field, tangent vector = directional derivative

$$\text{for all} \qquad f \in \mathcal{F}, t^a \in V_p, \qquad t(f) = t^a \nabla_a f$$

(6) Torsion free

$$\text{for all} \qquad f \in \mathcal{F}, \qquad \nabla_a \nabla_b f = \nabla_b \nabla_a f$$

(7) Inner product of two vectors remain unchanged during parallel-transport:

$$\nabla_a g_{bc} = 0 \tag{4.3}$$

Under these conditions, there exist a unique ∇_μ:

$$\nabla_\mu T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} = \partial_\mu T^{\sigma_1 \ldots \sigma_l}{}_{\eta_1 \ldots \eta_l} + \sum_i \Gamma^{\sigma_i}{}_{\mu\nu} T^{\sigma_1 \ldots \nu \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} \tag{4.4}$$

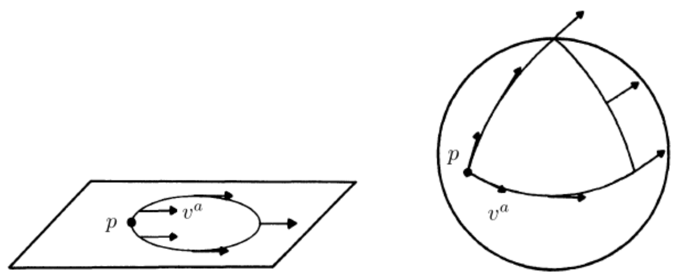
$$- \sum_j \Gamma^{\mu}{}_{\mu\eta_j} T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \nu \ldots \eta_l} \tag{4.5}$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\mu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \quad \text{(Christoffel symbol)} \tag{4.6}$$

• ∇_μ is a generalization of ∂_μ, when g is constant, $\Gamma^\rho_{\mu\nu} = 0$, $\nabla_\mu = \partial_\mu$.
• ∇_μ is usually not commutative. $\nabla_\mu \nabla_\nu T \neq \nabla_\nu \nabla_\mu T$.
• $\partial_\mu T$ usually do not produce a tensor, because ∇_μ does and $\nabla_\mu \neq \partial_\mu$ except special cases.

4.1.2 Parallel transport



• Parallel transport (P.T)
$\approx$ generalization in "curved space" the concept of "keeping the vector constant".
• Given a curve $x^\mu(s)$, the P.T of a vector v^ν along the curve in flat space is

$$\frac{dv^\nu}{ds} = \frac{\partial v^\nu}{\partial x^\mu} \frac{dx^\mu}{ds} = 0 \tag{4.7}$$

$\dfrac{\partial v^\nu}{\partial x^\mu}$ however is not a tensor in curved space, so we should generalize it:

$$t^\mu \nabla_\mu v^\nu = 0 \tag{4.8}$$

where $t^\mu = \dfrac{dx^\mu}{ds}$.
• P.T of a general tensor

$$t^\mu \nabla_\mu T^{\sigma_1 \ldots \sigma_l}{}_{\eta_1 \ldots \eta_l} = 0 \tag{4.9}$$

• This equation is an tensor equation. A tensor equation won't be changed under change of coordinate system.

$$T'^{\sigma'_1 \ldots \sigma'_k}{}_{\eta'_1 \ldots \eta'_l} = T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} \frac{\partial x'^{\sigma'_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x^{\eta_l}}{\partial x'^{\eta'_l}} = 0 \tag{4.10}$$

4.2 Curvature

"Curvature can be sensed by parallel transport".

4.2.1 Riemann curvature tensor

Consider $f \in \mathcal{F}, \omega \in V^*$, and

$$\begin{aligned}\nabla_\mu \nabla_\nu (f\omega_\sigma) &= \nabla_\mu (\nabla_\nu f \cdot \omega_\sigma + f \nabla_\nu \omega_\sigma) \\ &= (\nabla_\mu \nabla_\nu f)\omega_\sigma + \nabla_\nu f \cdot \nabla_\mu \omega_\sigma + \nabla_\mu f \nabla_\nu \omega_\sigma + f \nabla_\mu \nabla_\nu \omega_\sigma \end{aligned} \tag{4.11}$$

$$\nabla_\mu \nabla_\nu (f\omega_\sigma) - \nabla_\nu \nabla_\mu (f\omega_\sigma) = f(\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma) \tag{4.12}$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(f\omega_\sigma) = f(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\omega_\sigma \tag{4.13}$$

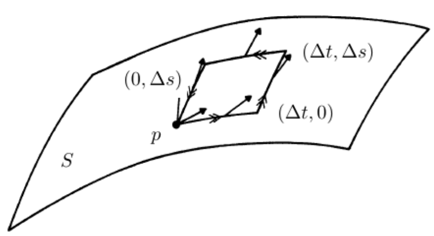
Now $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\omega_\sigma \Big|_p$ depend only on the value of ω_σ at p. Then $f\omega_\sigma \in V^*$, $f(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\omega_\sigma \in \mathcal{T}(0,3)$.
$\forall p \in \mathcal{M}$, $\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu : f\omega_\sigma \longrightarrow \mathcal{T}(0,3)$. Its action is of tensor type $(1,3)$.
There exist a tensor field $R_{\mu\nu\sigma}{}^{\eta}$, s.t. for ω_σ,

$$(\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma) = R_{\mu\nu\sigma}{}^{\eta} \omega_\eta \tag{4.14}$$

$R_{\mu\nu\sigma}{}^{\eta}$: Riemann tensor.
• $R_{\mu\nu\sigma}{}^{\eta}$ senses whether the manifold is curved.

4.2.2 How $R_{\mu\nu\sigma}{}^{\eta}$ is related to failure of a vector

Returning to its initial value after P.T. along closed curve, i.e., related to curvature.



Consider P.T. v^μ from point p. It is convinient if consider variation of $v^\mu \omega_\mu$, ω_μ being arbitrary dual vector field.

$$\begin{aligned}\delta_1 &= \Delta t \frac{\partial}{\partial t}(v^\mu \omega_\mu)\Big|_{(\frac{\Delta t}{2},0)} \\ &= \Delta t T^\nu \nabla_\nu (v^\nu \omega_\mu) \\ &= \Delta t T^\nu (\nabla_\nu \omega_\mu) v^\mu |_{(\frac{\Delta t}{2},0)}\end{aligned} \tag{4.15}$$

$T^\nu \equiv \dfrac{\partial x^\nu}{\partial t}$ is the tangent vector to curve at constant s.

$$\delta_1 + \delta_3 = \Delta t \Big\{ v^\mu T^\nu \nabla_\nu \omega_\mu|_{(\Delta t/2,0)} - v^\mu T^\nu \nabla_\nu \omega_\mu|_{(\Delta t/2,\Delta s)} \Big\} \longrightarrow \Delta t \mathcal{O}(\Delta s) \quad (4.16)$$

Similarly,

$$\delta_2 + \delta_4 \to \Delta s \mathcal{O}(\Delta t) \quad (4.17)$$

$\delta_1+\delta_3+\delta_2+\delta_4$ is at least a 2nd order infinitesimal. i.e., P.T. is path-independent at 1st order.

Indeed, we can evaluate the $\delta_1 + \delta_3$ to further accuracy.

$$\begin{aligned}
\delta_1 + \delta_3 &= \Delta t \Big\{ v^\mu T^\nu \nabla_\nu \omega_\mu|_{(\Delta t/2,0)} - v^\mu T^\nu \nabla_\mu \omega_\mu|_{(\Delta t/2,\Delta s)} \Big\} \\
&= -\Delta t \Delta s \frac{\partial}{\partial s} \Big(v^\mu T^\nu \nabla_\nu \omega_\mu|_{(\Delta t/2,\Delta s)} \Big) \\
&= -\Delta t \Delta s \Big[(T^\nu \nabla_\nu \omega_\mu)(S^\alpha \nabla_\alpha v^\mu) + v^\mu S^\alpha \nabla_\alpha (T^\nu \nabla_\nu \omega_\mu) \Big] \Big|_{(\Delta t/2,\Delta s/2)} \\
&= -\Delta t \Delta s v^\mu S^\alpha \nabla_\alpha (T^\nu \nabla_\nu \omega_\mu)|_{(\Delta t/2,\Delta s/2)} \\
&\approx -\Delta t \Delta s v^\mu T^\alpha S^\beta \nabla_\alpha (T^\nu \nabla_\nu \omega_\mu)|_p + \mathcal{O}(\Delta t^2)\mathcal{O}(\Delta t^2)
\end{aligned}$$

$$\begin{aligned}
\delta(v^\mu \omega_\mu) &= \Delta t \Delta s v^\mu [T^\alpha \nabla_\alpha (S^\nu \nabla_\nu \omega_\mu) - S^\alpha \nabla_\alpha (T^\nu \nabla_\nu \omega_\mu)] \\
&= \Delta t \Delta s v^\mu T^\alpha s^\beta (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\mu \\
&= \Delta t \Delta s v^\mu T^\alpha S^\beta R_{\alpha\beta\mu}{}^\nu \omega_\nu
\end{aligned} \quad (4.18)$$

Since ω_μ is arbitrary, the only possibility is

$$\delta(v^\nu) = \Delta t \Delta s v^\mu T^\alpha S^\beta R_{\alpha\beta\mu}{}^\nu \quad (4.19)$$

This shows that indeed $R_{\alpha\beta\mu}{}^\nu$ (Riemann tensor) is related to the path-dependence of parallel transport, which is further related to curvature.

4.2.3 Properties of Riemann tensor

(1) $R^\beta_{\mu\nu\alpha} = -R^\beta_{\nu\mu\beta}$, $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$
(2) $R^\beta_{[\mu\nu\alpha]} = 0$
(3) $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$
(4) $\nabla_{[\sigma} R^\beta_{\mu\nu]\alpha}$

Proof. (1) Definition, $\forall \omega_\mu$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\mu = R_{\alpha\beta\mu}{}^\nu \omega_\nu \quad (4.20)$$

(2) $\nabla_\mu \nabla_\nu \omega_\alpha = 0$.
(3) Before prove this, note a result similar to the introduction of Riemann tensor:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} = -\sum_{i=1}^{k} R_{\mu\nu\alpha}{}^{\sigma_i} T^{\sigma_1 \ldots \alpha \ldots \sigma_k}{}_{\eta_1 \ldots \eta_l} + \sum_{j=1}^{l} R_{\mu\nu\eta_i}{}^{\alpha} T^{\sigma_1 \ldots \sigma_k}{}_{\eta_1 \ldots \alpha \ldots \eta_l} \tag{4.21}$$

$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) g_{\alpha\beta} = R_{\mu\nu\alpha}{}^{\sigma} g_{\sigma\beta} + R_{\mu\nu\beta}{}^{\sigma} g_{\alpha\sigma} = R_{\mu\nu\alpha\beta} + R_{\mu\nu\beta\alpha} = 0 \tag{4.22}$$

(4)$\forall \omega_\alpha$,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \nabla_\alpha \omega_\beta = R_{\mu\nu\alpha}{}^{\sigma} \nabla_\sigma \omega_\beta + R_{\mu\nu\beta}{}^{\sigma} \nabla_\alpha \omega_\sigma \tag{4.23}$$

$$\nabla_\alpha (\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma) = \nabla_\alpha (R_{\mu\nu\sigma}{}^{\eta} \omega_\eta) = \omega_\eta \nabla_\alpha R_{\mu\nu\sigma}{}^{\eta} + R_{\mu\nu\sigma}{}^{\eta} \nabla_\alpha \omega_\eta \tag{4.24}$$

Autisymmetrize both equations for μ, ν, α, L.Hs becomes equal.

$$R^{\sigma}_{[\mu\nu\alpha]} \nabla_\nabla \omega_\beta + R^{\sigma}_{[\mu\nu|\beta|} \nabla_{\alpha]} \omega_\sigma = \omega_\eta \nabla_{[\alpha} R^{\eta}_{\mu\nu]\sigma} + R^{\eta}_{[\mu\nu|\sigma|} \nabla_{\alpha]} \omega_\eta \tag{4.25}$$

$$\nabla_{[\alpha} R^{\eta}_{\mu\nu]\sigma} = 0 \tag{4.26}$$

□

4.2.4 Ricci tensor, Ricci scalar, Einstein tensor, Weyl tensor

From properties (1),(2),(3),(4) of Riemann tensor, there are $\frac{n^2(n^2-1)}{2}$ independent components for n-dim manifold Riemann tensor. Decomposition into "trace part" and "trace free part":
(1)$R_{\mu\alpha} \equiv R_{\mu\nu\alpha}{}^{\nu}$ "trace part".
This is called Ricci tensor: symmetric $R_{\mu\alpha} = R_{\alpha\mu}$
From this, define Ricci scalar/curvature

$$R \equiv R^{\mu}_{\mu} \tag{4.27}$$

(2) Trace-free part

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{2}{n-2}(g_{\mu[\alpha} R_{\beta]\nu} - g_{\mu[\alpha} R_{\beta]\mu}) - \frac{2}{(n-1)(n-2)} R g_{\mu[\alpha} g_{\beta]\mu} \tag{4.28}$$

This is called Weyl tensor/conformal tensor.
(3) Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{4.29}$$

Property:$\nabla^\alpha G_{\mu\nu} = 0$

4.3 Geodesics

Intuatively, "straightest possible curve".

4.3.1 Geodesic equation

P.T. a vector v^μ along a curve with tangent T^ν

$$T^\nu \nabla_\nu v^\mu = 0 \tag{4.30}$$

Geodesic: A curve whose tangent vector is parallel propagated along itself:

$$T^\nu \nabla_\nu T^\mu = 0 \tag{4.31}$$

Supposing $T^\mu = T^\mu(t)$, i.e., the curve is parameterized by t, and using $\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\sigma} T^\sigma$, and $dT^\nu = \dfrac{dx^\nu}{dt}$, we get

$$\frac{dT^\mu}{dt} + \sum_{\sigma,\nu} \Gamma^\mu_{\sigma\nu} T^\sigma T^\nu = 0 \tag{4.32}$$

i.e., the curve is $x^\mu = x^\mu(t)$

$$\frac{d^2 x^\mu}{dt^2} + \sum_{\sigma,\nu} \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{dt} \frac{dx^\nu}{dt} = 0 \tag{4.33}$$

Eq(8.61), Eq(8.64), Eq(4.33) are the **Geodesic equations.**
• It is a system of second order ordinary differential equations (ODEs). Mathematically, a initial value problem of $x^\mu(t)$. Theory of ODE solution uniqueness tells us:

There always exists a unique solution for any given initial $x^\mu(t_0)$ and $dx^\mu/dt|_{t=t_0}$. I.e., given any $p \in C \subset M$ and a tangent T_p at p, there exists a unique geodesic passing p with tangent T_p.

4.3.2 Proper length or proper time along the geodesics

• A vector T^μ is called $\begin{cases} \text{timelike} & g_{\mu\nu} T^\mu T^\nu < 0 \\ \text{null} & g_{\mu\nu} T^\mu T^\nu = 0 \\ \text{spacelike} & g_{\mu\nu} T^\mu T^\nu > 0 \end{cases}$, $g_{\mu\nu} T^\mu T^\nu$ is called norm of T^μ. $g_{\mu\nu} T^\mu S^\nu$ is called inner product of T^μ and S^ν.
• A vector pair can not change their inner product during P.T. A vector can not change its norm during P.T.

Proof. For tangent t^μ of any curve,

$$\begin{aligned}
&t^\mu\nabla_\mu(g_{\sigma\eta}T^\sigma S^\eta)\\
&=(t^\mu\nabla_\mu T^\sigma)g_{\sigma\eta}S^\eta+(t^\mu\nabla_\mu S^\eta g_{\sigma\eta}T^\sigma)+t^\mu(\nabla_\mu g_{\sigma\eta}T^\sigma S^\eta)\\
&=0+0+0=0
\end{aligned}\tag{4.34}$$

t^μ is arbitrary $\Rightarrow g_{\sigma\eta}T^\sigma S^\eta=$ constant during P.T.

□

• Then, a geodesic (whose tangent is P.T. along itself), its tangent vectors at all points are either timelike, null, spacelike; but not change its charcter during P.T.

$$\text{A geodesic's(or curve) tangents}\begin{cases}\text{timelike}\Rightarrow\text{timelike geodesic(curve)}\\ \text{null}\Rightarrow\text{null geodesic(curve)}\\ \text{spacelike}\Rightarrow\text{spacelike geodesic(curve)}\end{cases}$$

For timelike geodesic/curve, define proper time

$$\tau=\int_{t=t_i}^{t=t_f}(-g_{\mu\nu}T^\mu T^\nu)^{\frac{1}{2}}dt\tag{4.35}$$

For spacelike geodesics/curve, define proper length

$$l=\int_{t=t_i}^{t=t_f}(g_{\mu\nu}T^\mu T^\nu)^{\frac{1}{2}}dt\tag{4.36}$$

or combined

$$l=\int_{t_i}^{t_f}(|g_{\mu\nu}T^\mu T^\nu|)^{\frac{1}{2}}dt\tag{4.37}$$

4.3.3 Properties of (timelike/null/spacelike) geodesics

(1) Proper time or proper length is independent of the parameterization of the curve. Consider $x^\mu\longrightarrow x^\mu(t(s))=x^\mu(s)$

$$\begin{aligned}
\tau&=\int(-g_{\mu\nu}T^\mu T^\nu)^{\frac{1}{2}}dt=\int(-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt})^{\frac{1}{2}}dt\\
&=\int(-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds})^{\frac{1}{2}}\frac{ds}{dt}\cdot dt\\
&=\int(-g_{\mu\nu}s^\mu s^\nu)^{\frac{1}{2}}ds=\tau_s
\end{aligned}\tag{4.38}$$

Essentially, reparameterization is a change of variable for a definite integral.

4.3.4 A globally extreme curve (shortest length or greatest proper time)

Connecting two points, if exists, there must be a (spacelike or timelike) geodesic. A spacelike/timelike geodesic is at least a local extreme.

Proof. Consider a spacelike curve, fixed end points $t = a$, $t = b$,

$$l = \int_a^b (\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt})^{\frac{1}{2}} dt \tag{4.39}$$

$$\delta l = \int_a^b [\sum_{\mu\nu} g_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu dt}{]}^{-\frac{1}{2}} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{dx^\alpha}{dt}\frac{d(\sigma x^\beta)}{dt} + \frac{1}{2}\sum_\sigma \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} dt \tag{4.40}$$

l will not be changed by reparameterization, therefore we can set $\sum_{\mu,\nu} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 1$

$$\begin{aligned} \delta l = &\int_a^b \sum_{\alpha,\beta} g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{d\delta x^\beta}{dt} + \frac{1}{2}\sum_\sigma \frac{g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma \frac{dx^\alpha}{dt}\frac{x^\beta}{dt} dt \\ &- \sum_\alpha g_{\alpha\beta}\frac{d^2x^\alpha}{dt^2} - \sum_{\alpha\lambda}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\frac{dx^\lambda}{dt}\frac{x^\alpha}{dt} + \frac{1}{2}\sum_{\alpha\lambda}\frac{\partial g_{\alpha\lambda}}{\partial x^\beta}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt} = 0 \\ &- g_{\alpha\beta}\frac{d^2x^\gamma}{dt^2} + \frac{1}{2}[\partial_\beta g_{\alpha\beta} - \partial_\lambda g_{\alpha\beta} - \partial_\alpha g_{\lambda\beta}]\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt} = 0 \\ &- \frac{d^2x^\gamma}{dt^2} + \frac{1}{2}g^{\beta\gamma}[\partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta} - \partial_\alpha g_{\lambda\beta}]\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt} = 0 \end{aligned} \tag{4.41}$$

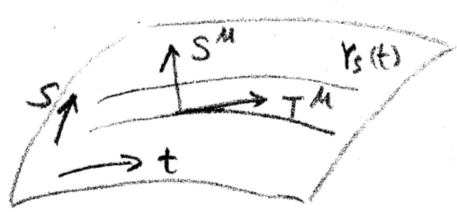
Geodesic equation

$$\frac{d^2x^\gamma}{dt^2} + \Gamma^\gamma_{\alpha\gamma}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt} = 0 \tag{4.42}$$

□

4.4 Collection of geodesics

In flat manifold, initially parallel geodesics remain parallel forever. In curved manifold, they will not necessarily do this. we study how geodesics deviate from each other.



• To be concerete, consider a one-parameter family of geodesics, $\gamma_s(t)$. I.e., for each $s \in \mathbb{R}$

$$\gamma_s(t) : t \longrightarrow C \in \mathcal{M} \tag{4.43}$$

Here t is the affine parameter.
Then these geodesics define a $1+1$ dimensional surface embedded in $\mathcal{M}$. Its coordinates can be chosen as s and t:

$$\mathcal{M} = x^\mu(s,t) \tag{4.44}$$

Two natural vector fields

$$T^\mu = \frac{\partial x^\mu}{\partial t} \qquad \text{tangent vector} \tag{4.45}$$

$$S^\mu \equiv \frac{\partial x^\mu}{\partial s} \qquad \text{“deviation vector”} \tag{4.46}$$

• Then how fast the s^μ changes can characterize how fast geodesics deviate from each other. Define “relative velocity of geodesics”

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu \tag{4.47}$$

Further more, how fast this “velocity” change becomes “acceleration”.
Define “relative accelaration of geodesics”

$$a^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu \tag{4.48}$$

The names are just names, but these quantities are well defined.
• We would like to show a^μ is related to the curvature of the manifold.

$$\begin{aligned}
a^\mu &= T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) && (4.49)\\
&= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)\\
&= (T^\rho \nabla_\rho s^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu\\
&= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu_{\nu\rho\sigma} T^\nu)\\
&= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + s^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) + R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu\\
&= R^\mu_{\nu\rho\sigma} T^\nu T_\rho S^\sigma && (4.50)
\end{aligned}$$

commutator of covariant derivatives and geodesic condition $T^\rho \nabla_\rho T^\mu = 0$ are used.

Therefore, geodesics will accelerate towards or away from each other if and only if $R^\mu_{\nu\alpha\beta} \neq 0$. Initially parallel geodesics will not be parallel again if and only if $R^\nu_{\nu\alpha\beta} \neq 0$.

Chapter 5

Motivation and special relativity

5.1 Short review of S.R.

Now we motivate the introduction of general relativity, during this we also do a short review of S.R.

• Why we study tensor calculus?

All physical laws can be expressed as a tensor equation.

All physical measurements are either scalar or components of vectors, tensors.

• Physical laws should not depend on the frame or any particular vector/tensor fields.

I.e., laws formulated in different frames should predict the same physics.

More generally, the metric is the only quantity associated with spacetime that will appear in physics laws.(General covariance principle)

• In prerelativity physics, laws also follow special covariance principle, physical laws (written in components form) remain unchanged under metric rotation and translation; plus time reversal and space parity.

5.2 Special Relativity

5.2.1 The spacetime in S.R. is an $\mathbb{R}^4$ manifold

The mapping *spacetime* $\longrightarrow \mathbb{R}^4$ is called a global inertial coordinate system(I.C.S).

The infinitesimal spacetime interval is

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{5.1}$$

Remember the norm(or “interval”) between two points on a manifold is

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.2}$$

This suggest metric in S.R. is

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1,) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{5.3}$$

$\Rightarrow$ Christoffal symbol $\Gamma^\mu_{\nu\sigma} = 0$;
$\Rightarrow$ Covariant derivative $\nabla_\mu = \partial_\mu$;
Spacetime in SR is an 4-d manifold with metric $\eta_{\mu\nu}$. This spacetime is called Minkowski space.

5.2.2 Pioncare transforms

S.R. also asserts that the speed of light in vaccum in any inertial coordinate system is the same.
Combing with the isotropic property(or assumption) of spacetime, one can prove that the spaceetime interval is invariant in different I.C.S. Also, from the relativity principle, I.C.S. can only be connected by linear transforms.
Now linear transforms should be

$$[X'^\mu] = \Lambda[X^\mu] + [A^\mu] \tag{5.4}$$

(1)$[A^\mu]$ part correspond to translation:

$$x'^\mu = x^\mu + a^\mu \tag{5.5}$$

(2) $\Lambda[x^\mu]$ part correspond to “generalized rotation”.
To have spacetime intervals invariant

$$\begin{aligned} (ds)^2 = [dx^\mu]^T[\eta_{\mu\nu}][dx^\nu] &= [dx'^\mu]^T \eta_{\mu\nu}[dx'^\nu] \\ &= [dx^\mu]^T \Lambda^T [\eta_{\mu\nu}]\Lambda[dx^\nu] \end{aligned} \tag{5.6}$$

$$[\eta_{\mu\nu}] = \Lambda^T[\eta_{\mu\nu}]\Lambda \tag{5.7}$$

$$\eta_{\rho\sigma} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} \tag{5.8}$$

$[\Lambda]$ called Lorentz transforms.

First kind Λ: conventional rotations

$$[\Lambda^\mu_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.9}$$

Rotation in x-y plane by angle θ.

Second kind Λ boost:

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.10}$$

Boost along X-direction.
Why this is a boost?

Consider the $[X'^\mu] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ point in the $[x^\mu]$ coordinate system

$$\begin{cases} t' = t\cosh\phi - x\sinh\phi \\ x' = -t\sinh\phi + x\cosh\phi \end{cases}$$

The $t' = 0, x'^1 = x'^2 = x'^3 = 0$ point correspond to (t, x^1, x^2, x^3) point s.t

$$\frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi \xrightarrow{\phi=\tanh^{-1} v} v \tag{5.11}$$

So we went from a I.C.S. of velocity zero to velocity $v \Rightarrow$ *boost*

$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases}$$

$\gamma = \dfrac{1}{\sqrt{1-v^2}}$. Length contraction and time dilation.
There are 6 independent rotations and boosts. They form a proper Lorentz group, SO(3,1), non-abelian.

Proper Lorentz transforms, together with 4 translations, from a ten parameter group, called poincare group. Non-abelian either.

Third kind Λ: discrete transform

$$[T] = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{time reverse} \tag{5.12}$$

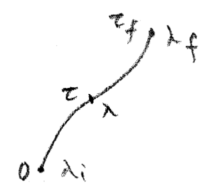
$$[T] = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \qquad \text{parity transform} \tag{5.13}$$

The proper Lorentz transform plus time reverse and parity forms a full Lorentz group, $O(3,1)$.

5.2.3 Physical laws in S.R.

All quantities and laws has to be written as tensors to garantie invariance in different I.C.S.

• 4-velocity:



A timelike curve C parameterized by its proper time τ.

4 velocity $u^\mu \equiv \dfrac{dx^\mu(\tau)}{d\tau}$

Property: $u^\mu u_\mu = -1$.

Proper time is defined as for any parameterization $x^\mu(\lambda)$

$$\tau = \int_{\lambda_i}^{\lambda} (-g_{\mu\nu} T^\mu T^\nu)^{\frac{1}{2}} d\lambda \tag{5.14}$$

where $T^\mu = \dfrac{dx^\mu(\lambda)}{d\lambda}$.

• 4 momentum: material particles have an attribute called "real mass" m.

4-momentum:

$$p^\mu = m u^\mu \tag{5.15}$$

• Energy and energy-momentum tensor

Energy measured by a moving observer with 4-velocity v^μ of an particle with 4-velocity u^μ is defined as

$$E = -p_\mu v^\mu = -m g_{\mu\nu} u^\mu v^\nu \tag{5.16}$$

when observer is at rest w.r.t particle, $v^\mu = u^\nu$

$$E = mc^2, \qquad \text{where } c = 1 \tag{5.17}$$

Generalize this idea, the energy-momentum tensor of a contimass matter(stress-energy tensor) is denoted as $T_{\mu\nu}$.

$$T_{\mu\nu} v^\mu v^\nu \Rightarrow \text{energy density}$$

i.e., the mass energy per unit volume measured by an v^μ observer.

Conservation of matter: $\partial_\mu T^{\mu\nu} = 0$.

5.2.4 Different matter

• Perfect fluid: a fluid that is isotropic in its rest frame and having no stress.

$$\begin{aligned} T_{\mu\nu} &= \rho u_\mu u_\nu + p(\eta_{\mu\nu} + u_\mu u_\nu) \\ &= (\rho + p) u_\mu u_\nu + p\eta_{\mu\nu} \end{aligned} \tag{5.18}$$

In rest frame:

$$[T_{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} \tag{5.19}$$

For pressureless dust in rest frame:

$$[T_{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \tag{5.20}$$

Conservation:

$$\partial^\mu T_{\mu\nu} = 0 \tag{5.21}$$

ρ: energy density in rest frame.
p: pressure in rest frame.
u^μ: 4-velocity of fluid.

• In the limit of nonrelativistic, $p \ll \rho$, $u^\mu = (1, \vec{v})$, $|\vec{v}|\dfrac{dp}{dt} \ll |\vec{\nabla}p|$, derive the Euler's equations of fluids.

• Electromagnetic field

$$T_{\mu\nu} = \frac{1}{4\pi}F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{5.22}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called field strength, A_μ is the vector potential.
Electric vector field $E_\mu = F_{\mu\nu}v^\nu$.
Magnetic vector field $B_\mu = -\frac{1}{2}\epsilon_{\mu\alpha}{}^{\beta\gamma}F_{\beta\gamma}v^\alpha$, ϵ is the totally anti-symmetric tensor with $\epsilon_{0123} = 1$.
Maxwell's equation

$$\begin{cases} \partial^\mu F_{\mu\nu} = -4\pi j_\nu \\ \partial_{[\mu}F_{\alpha\beta]} = 0 \end{cases} \qquad j_\nu \text{ is the electric charge density current.}$$

Lorentz force law on a particle with 4-velocity u^μ:

$$u^\mu \partial_\mu u^\nu = \frac{q}{m}F^{\nu}_{\mu}u^\mu \tag{5.23}$$

• Scalar field described by Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) \tag{5.24}$$

The energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\mu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi + m^2\phi^2) \tag{5.25}$$

Prove that for any observer, the E.M.T for perfect fluid, electromagnetic field and scalar field satisfies the condition

$$T_{\mu\nu}v^\mu v^\nu \geq 0 \tag{5.26}$$

This is called (weak) energy condition.

Chapter 6

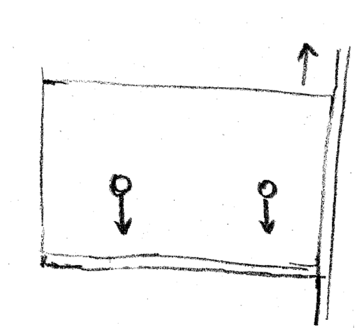
General Relativity and Einstein Equation

6.1 Motivation

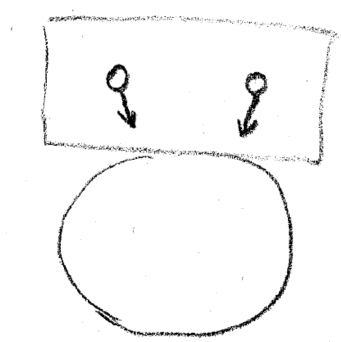
We start from basic principles and attempt to argue that they lead naturally to an almost unique physical theory.
• Weak equivalence principle(WEP)

$$\vec{f} = m_i \vec{a} \qquad m_i : \text{inertial mass}$$
$$\vec{f}_g = m_g \vec{\nabla}\phi$$
$$m_g : \text{gravitational mass} \quad \phi : \text{gravitational potential}$$

WEP: $m_i \neq m_g$ is the same for anybody.



• Einstein E.P.



⇒ The above box should be small for the argument to work.
⇒ After S.R., the equivalence of "mass" becomes less meaningful. After all, mass can be converted to other format.
This motivated Einstein for his E.E.P.:(the principle of equivalence)

In small enough regions, the law of physical reduces to those of S.R. All bodies are influenced by gravity, and all bodies fall precisely the same way in a gravitational field.
E.E.P⇒ Spacetime should be considered as a curved manifold.

↓ No global inertial frame because gravity is inescapable. The establishment of global inertial frame in S.R. implicitly assumed the existence of a force-free reference. Now this is impossible. "The acceleration due to gravity" becomes an imprecise statement, because we don't have a frame for measurement.
↓ Instead we say bodies under influence of gravity (only), do "free falling" (along their geodesics in a curved spacetime).
↓ Inertial frames have to become "local". When the region is small enough, we expect the influence of gravity on the "box(frame)" is universal. S.R. should be restored here.
↓ The best we can do is to associate a "local inertial frame" with each particle moving freely in gravity (free falling).

Einstein then tried to find an equation that describes the influence of gravity on matter. [Note the E.E.P. is a postulate: it can not be proven but be falsified.]

6.2 Einstein equation

We start from an equation in classical mechanics, which connects the spacetime geometry with matter distribution, and then use the above principles and intuitions as guidelines. Study the description of tidal acceleration in Newtonian mechanics and G.R. I.e., Consider the free fall of two test particles:

In N.M., the free falling is described by

$$\left(\frac{d^2x_i}{dt^2}\right)_P = -\left(\frac{\partial\phi}{\partial x^i}\right)_P, \qquad \left(\frac{d^2x_j}{dt^2}\right)_Q = -\left(\frac{\partial\phi}{\partial x^j}\right)_Q \tag{6.1}$$

ϕ: gravitational potential.

Then the separation of P and Q, changes as free falls., $\vec{x} = (x_j)_P - (x_j)_Q$. When the separation is itself small

$$\frac{d^2\vec{x}_j}{dt^2} = -\left(\frac{\partial^2\phi}{\partial x^j\partial x^k}\right)\vec{x}_k \tag{6.2}$$

Taylor expansion can prove this.
I.e., the "relative acceleration of geodesics"

$$a^\mu = -(\partial_\beta\partial^\mu\phi)x^\beta \tag{6.3}$$

In G.R./differential geometry,

$$a^\mu = -R_{\alpha\beta\gamma}{}^{\mu}v^\alpha v^\gamma x^\beta \tag{6.4}$$

• Guess $R_{\alpha\beta\gamma}{}^{\mu}v^\alpha v^\gamma \longleftrightarrow \partial_\beta\partial^\mu\phi$.

$$\nabla^2\phi = 4\pi\rho \qquad \text{Poisson's equation} \tag{6.5}$$

ρ: mass(energy) density of matter.

$$\rho \longleftrightarrow T_{\alpha\gamma}v^\alpha v^\gamma \tag{6.6}$$

$T_{\alpha\gamma}$:E.M. tensor
v^μ: velocity of free falling fluid
• Suggest:

$$R_{\alpha\mu\gamma}{}^{\mu}v^\alpha v^\gamma = 4\pi T_{\alpha\gamma}v^\alpha v^\gamma \tag{6.7}$$
$$R_{\alpha\gamma} = 4\pi T_{\alpha\gamma} \tag{6.8}$$

This was first postulated by Einstein.
• However from Bianchi identity,

$$\nabla^{\alpha}(R_{\alpha\gamma} - \frac{1}{2}g_{\alpha\gamma}R) = 0. \tag{6.9}$$

From EM tensor conservation,

$$\nabla^{\alpha}T_{\alpha\gamma} = 0 \tag{6.10}$$

so applying ∇^{α} to Eq(6.8), we get

$$\nabla_{\alpha}R = 0 \tag{6.11}$$
$$R = const \tag{6.12}$$
$$T^{\mu}_{\mu} = const \quad \text{for any matter} \tag{6.13}$$

This is apparently unphysical.
• Now we want a relation satisfies Eq (6.10) and Eq (6.7). Such and equation is found by Einstein in 1915,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{6.14}$$

The LHS tensor is actually named after Einstein, the Einstein tensor.
(1) This is the celebrated Einstein equation.
(2) From now on, we shift our focus of the course to the solution of this equation.
• First let us see that Eq (6.7) is indeed satisfied.
From

$$R = -8\pi T \tag{6.15}$$
$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{6.16}$$

when observer is roughly at rest with fluid $v^{\mu}v_{\mu} = -1$

$$T \approx -\rho = -T_{\alpha\beta}v^{\alpha}v^{\beta} \tag{6.17}$$

Then,

$$\begin{aligned} R_{\mu\nu}v^{\mu}v^{\nu} &= 8\pi(T_{\mu\nu}v^{\mu}v^{\nu}) + 4\pi g_{\mu\nu}T_{\alpha\beta}v^{\alpha}v^{\beta}v^{\mu}v^{\nu} \\ &= 4\pi T_{\mu\nu}v^{\mu}v^{\nu} \end{aligned} \tag{6.18}$$

This recovers Eq(6.7).

6.3 Physical laws in G.R.

• general covariance
• locally reduce to S.R.

(1) In G.R., physical quantities are still described by tensors.
Motions of particle are described by timelike curves.
Perfect fluid velocity u^μ.
e.m.t, $T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$, E.m. field:$F_{\mu\nu}$.
(2) Physical laws are still described by tensor equations.
"Two principles requires equation being invariant under any coordinate transform."
Two rules should be applied to equations in S.R.
(a)

$$\begin{array}{ccc} \eta_{\mu\nu} \longrightarrow & & g_{\mu\nu} \\ \downarrow & & \downarrow \\ dig(-1,1,1,1) & & general \end{array}$$

(b) $\partial_\mu \longrightarrow \nabla_\mu$ "minimal substitation"
A particle: momentum $p^\mu = m u^\mu$
Perfect fluid: E.O.M. $\nabla^\mu T_{\mu\nu} = 0$

$$\Rightarrow \begin{cases} u^\mu \nabla_\mu \rho + (\rho + p)\nabla^\mu u_\mu = 0 \\ (p+\rho) u^\mu \nabla_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu)\nabla^\mu p = 0 \end{cases}$$

Scalar field:

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla_\sigma \phi \nabla^\sigma \phi + m^2 \phi^2) \tag{6.19}$$

Electromagnetic field

$$T_{\mu\nu} = \frac{1}{4}\{F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\} \tag{6.20}$$

Maxwell's equation

$$\nabla^\mu F_{\mu\nu} = -4\pi j_\nu \tag{6.21}$$

(3) However, these two rules "$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$, $\partial_\mu \longrightarrow \nabla_\mu$" are only guide lines. It did not recover terms that will only appear due to curvature. For example, Mexwell's equations for vector gauge field A_μ(in the Lorentz gauge)

$$\nabla^\mu \nabla_\mu A_\nu - R^\mu_\nu A_\mu = -4\pi j_\nu \tag{6.22}$$

However, in S.R., the corresponding equations was

$$\partial^\mu \partial_\mu A_\nu = -4\pi j_\nu \tag{6.23}$$

$$\nabla^\mu \nabla_\mu A_\nu = -4\pi j_\nu \tag{6.24}$$

If "minimal substitution" was used, the $-R^\mu_\nu A_\mu$ therm won't be constructed. However, Eq(6.22) is favors over Eq(6.23) because it satisfies current conservation $\nabla_\nu j^\nu = 0$. So, the true physical equation really is determined from physics. (In this case, the one with $-R^\mu_\nu A_\mu$), not the "minimal substitution" rules.

However, they are good guide lines. In our course, all equations will be the one obtained from physical consideration.

Chapter 7

Linear gravity

• G.R. is a new description of gravity. It should reduce to Newton's gravity in where it works: weak gravity and low speed.

7.1 Linearize G.R.

Consider metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) \tag{7.1}$$

$|\gamma_{\mu\nu}| \ll 1$, then,

$$g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \tag{7.2}$$

the Christoffel symbol

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu \gamma_{\nu\beta} + \partial_\nu \gamma_{\mu\beta} - \partial_\beta \gamma_{\mu\nu}) \tag{7.3}$$

Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}_{\alpha\nu} \\ &= \partial^\alpha \partial_{(\nu}\gamma_{\mu)\alpha} - \frac{1}{2}\partial^\alpha \partial_\alpha \gamma_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu \gamma \end{aligned} \tag{7.4}$$

here $\gamma = \gamma^{\alpha}_{\alpha}$.
Einstein tensor

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\eta\nu} R \\ &= \partial^\alpha \partial_{(\nu}\gamma_{\mu)\alpha} - \frac{1}{2}\partial^\alpha \partial_\alpha \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\partial^\alpha \partial^\beta \gamma_{\alpha\beta} - \partial^\alpha \partial_\alpha \gamma) \end{aligned} \tag{7.5}$$

Let us define $\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma$, then Einstein Equation,

$$G_{\mu\nu} = -\frac{1}{2}\partial^\alpha \partial_\alpha \bar{\gamma}_{\mu\nu} + \partial^\alpha \partial_{(\nu}\bar{\gamma}_{\mu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha \partial^\beta \bar{\gamma}_{\alpha\beta} = 8\pi T_{\mu\nu} \tag{7.6}$$

Let us do an infinitesimal gauge transform with vector field ξ_α,

$$x'^\alpha = x^\alpha + \xi^\alpha \qquad |\xi^\alpha / x^\alpha| \ll 1 \tag{7.7}$$

then,

$$\begin{aligned} g'^{\mu\nu} &= g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \\ &= (\eta_{\alpha\beta} + \gamma_{\alpha\beta})(\delta^\alpha_\mu + \partial_\mu \xi^\alpha)(\delta^\beta_\nu + \partial_\nu \xi^\beta) & (7.8) \\ \eta_{\mu\nu} + \gamma'_{\mu\nu} &= \eta_{\mu\nu} + \gamma_{\mu\nu} + \partial_\nu \xi_\mu + \partial_\mu \xi_\nu + \mathcal{O}(\xi^2) & (7.9) \\ \gamma'_{\mu\nu} &= \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu & (7.10) \end{aligned}$$

and,

$$\begin{aligned} \gamma' = \gamma'^\alpha_\alpha &= \gamma^\alpha_\alpha + 2\partial^\alpha \xi_\alpha = \gamma + 2\partial^\alpha \xi_\alpha & (7.11) \\ \bar{\gamma}'_{\mu\nu} &= \gamma'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma' \\ &= \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(\gamma + 2\partial^\alpha \xi_\alpha) \\ &= \bar{\gamma}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\alpha \xi_\alpha & (7.12) \\ \partial^\nu \bar{\gamma}'_{\mu\nu} &= \partial^\nu \bar{\gamma}_{\mu\nu} + \partial^\nu \partial_\mu \xi_\nu + \partial^\nu \partial_\nu \xi_\mu - \partial_\mu \partial^\alpha \xi_\alpha \\ &= \partial^\nu \bar{\gamma}_{\mu\nu} + \partial^\nu \partial_\nu \xi_\mu & (7.13) \end{aligned}$$

Since ξ^α is the gauge, we have the freedom to choose it s.t.,

$$\partial^\nu \bar{\gamma}_{\mu\nu} = -\partial^\nu \partial_\nu \xi_\mu \tag{7.14}$$

then, $\partial^\nu \bar{\gamma}'_{\mu\nu} = 0$.(This is an analog of the "Lorentz gauge" condition in $E\&M$ theory.)

Now the E.E under the gauge transform becomes

$$-\frac{1}{2}\partial^\alpha \partial_\alpha \bar{\gamma}'_{\mu\nu} = 8\pi T'_{\mu\nu} \tag{7.15}$$

Dropping the $'$ for the simplicity sake,

$$\partial^\alpha \partial_\alpha \bar{\gamma}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad (L.E.E) \tag{7.16}$$

7.2 The Newtonian limit

• In the Newtonian limit, the fluids should have

$$\begin{aligned} \rho &\gg p \qquad \text{(rest energy dominates)} \\ \text{4-velocity} u^\mu &= (1,0,0,0) \quad \text{(slow speed condition)} & (7.17) \end{aligned}$$

then their e.m.t becomes

$$T_{\mu\nu} \approx \rho u_\mu u_\nu \tag{7.18}$$

• More generally, for other matter form in the Newtonian limit, we assume

$$[T_{\mu\nu}] = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right] \tag{7.19}$$

the ignorance of "0i" $i = 1,2,3$ components means that the velocity is small, the ignorance of "ij" $i = 1,2,3$ components means stress is small.
• We therefore expect the metric to vary slowly.
We seek solutions with $\partial_0 \gamma_{\mu\nu} = 0$.
Then the L.E.E becomes

$$\nabla^2 \bar{\gamma}_{00} = -16\pi\rho \tag{7.20}$$
$$\nabla^2 \bar{\gamma}_{ij} = 0 \tag{7.21}$$
$$\tag{7.22}$$

where $\nabla^2 = \partial^1\partial_1 + \partial^2\partial_2 + \partial^3\partial_3$ is the Laplacian operator on space index. The solution to Eq (7.21) and that satisfy the good asymptotic flatness is

$$\bar{\gamma}_{ij} \tag{7.23}$$

The solution to Eq (7.20) is the following

$$\bar{\gamma}_{00} = -4\phi \tag{7.24}$$

where ϕ satisfies

$$\nabla^2 \phi = 4\pi\rho \tag{7.25}$$

Therefore $\bar{\gamma}_{\mu\nu} = -4\phi t_\mu t_\nu$, $\bar{\gamma} = 4\phi$. Then the perturbative metric becomes

$$\begin{aligned} \gamma_{\mu\nu} &= \bar{\gamma}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\gamma \\ &= \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma} \\ &= -(4t_\mu t_\nu + 2\eta_{\mu\nu})\phi \end{aligned} \tag{7.26}$$

The motion of a test particle is given by its geodesic

$$\frac{d^2x^\mu}{d\tau^2} + \sum_{\rho,\sigma} \Gamma^\mu_{\rho\sigma}(\frac{dx^\rho}{d\tau})(\frac{dx^\sigma}{d\tau}) = 0 \tag{7.27}$$

Now in Newtonian limit, particle is slow moving

$$[dx^\mu/d\tau] = [1,0,0,0] \tag{7.28}$$
$$\tau \approx t \tag{7.29}$$

Then the geodesics Eq.

$$\frac{d^2x^\mu}{dt^2} = \Gamma^\mu_{00} \tag{7.30}$$

For $\mu = 1, 2, 3$, using $\gamma_{\mu\nu}$ we get after neglecting time derivative

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\frac{\partial \gamma_{00}}{\partial x^{\mu}} = \frac{\partial \phi}{\partial x^{\mu}} \tag{7.31}$$

Finally, we get

$$\frac{d^2 x^{\mu}}{dt^2} = -\partial_{\mu}\phi \qquad \mu = 1, 2, 3 \tag{7.32}$$

i.e.

$$\vec{a} = -\vec{\nabla}\phi \tag{7.33}$$

We then recognize that

- ϕ is the gravitational potential
- the Eq (7.33) equation is the Newton's law under gravity
- the Eq (7.25) is the Poisson equation

Newtonian limit is recovered.

7.3 Gravitational Radiation

7.3.1 The Coulomb gauge

The linearized E.E. in a sourcefree spacetime is

$$\partial^{\alpha}\partial_{\alpha}\bar{\gamma}_{\mu\nu} = 0 \qquad (L.E.E) \tag{7.34}$$

It should also satisfy the Lorentz gauge condition

$$\partial^{\mu}\bar{\gamma}_{\mu\nu} = 0 \qquad (Lor.cond) \tag{7.35}$$

In obtaining these, we made the gauge transform

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}^{\text{(original)}} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} \tag{7.36}$$

and ξ^{μ} should satisfy

$$\partial^{\mu}\partial_{\mu}\xi_{\nu} = 0 \tag{7.37}$$

Now we would like to show that there exist wave solutions to the L.E.E. The key is Lorentz gauge condition does not fix the gauge completely. In addition to Eq(7.37), we can set the condition for ξ^{μ} at a initial surface $t = t_0$:

$$2\Big(-\frac{\partial \xi_0}{\partial t} + \vec{\nabla}\cdot\vec{\xi}\Big) = -\gamma \tag{7.38}$$

$$2\Big(-\vec{\nabla}^2\xi_0 + \vec{\nabla}\cdot\Big(\frac{\partial \vec{\xi}}{\partial t}\Big)\Big) = -\frac{\partial \gamma}{\partial t} \tag{7.39}$$

$$\frac{\partial \xi_{\mu}}{\partial t} + \frac{\partial \xi_0}{\partial x^{\mu}} = -\gamma_{0\mu} \quad (\mu = 1, 2, 3) \tag{7.40}$$

$$\vec{\nabla}^2\xi_{\mu} + \frac{\partial}{\partial x^{\mu}}\Big(\frac{\partial \xi_0}{\partial t}\Big) = -\frac{\partial \gamma_{0\mu}}{\partial t} \quad (\mu = 1, 2, 3) \tag{7.41}$$

This is an first order ODE system of $\{\partial_\mu \xi_\nu, \mu, \nu = 0, 1, 2, 3\}$.

We assert that with these conditions on ξ_μ,

$$\gamma = 0 \tag{7.42}$$
$$\gamma_{0\mu} = 0 \quad (\mu = 1, 2, 3) \tag{7.43}$$

can be achieved throughout the source free spacetime.
Then substitute back into (Lorentz cond)

$$\frac{\partial \gamma_{00}}{\partial t} = 0 \tag{7.44}$$

and into (L.E.E)

$$\vec{\nabla}^2 \gamma_{00} = 0 \tag{7.45}$$

The only sensable solution then is $\gamma_{00} = 0$.(constant can be gauged away). In short, the following can be achieved by just manipulating the gauge

$$\gamma = 0 \qquad (gc2) \tag{7.46}$$
$$\gamma_{0\mu} = 0 \qquad (gc3) \tag{7.47}$$

The combination of (Lorentz gauge) and (gc2),(gc3) forms the radiation (Coulomb) gauge condition.

Now we seek plane wave solutions of the L.E.E. of the form

$$\gamma_{\mu\nu} = H_{\mu\nu} exp(ikx^\alpha) \tag{7.48}$$

where $H_{\mu\nu}$ = constant field.
Then the LEE's solution exists if and only if

$$k^\mu k_\mu = 0 \tag{7.49}$$

the coulomb gauge conditoin

$$k^\mu H_{\mu\nu} = 0 \tag{7.50}$$
$$H_{0\nu} = 0 \tag{7.51}$$
$$H^\mu_\mu = 0 \tag{7.52}$$

There are $4+4+1-1$ equations (Eq(7.50),Eq(7.51) have 1 redudancy: $H_{0\nu}k^\nu = 0$), and there are 10 free components of $H_{\mu\nu}$. Therefore Eq(7.50)-Eq(7.52) suggest that the gravitational wave have 2 independent($= 10 - 8$) polarization states. The wave is still waiting to be detected.

Chapter 8

The Schwarzschild Solution

Now we would like to put G.R. to work:
  to predict something beyond the Newtonian gravity.
Mathematically, put E.E. to work:
  wherever gravity is important & manifest & $T_{\mu\nu}$ is not too complicated.
$\Rightarrow$ Exterior space:
  (a)gravity of our solar system $\Rightarrow$ Schwarzsbhicd solution
  (b)or some other exotic objects $\Rightarrow$ e.g. black holes
  (c)or even larger space. e.g. our universe $\Rightarrow$ cosmology
We do the study according to $(a) \rightarrow (c) \rightarrow (b)$ order

8.1 The spherecially symmetric metric

Apparently, the E.E will be much simpler if the metric has some symmetry.

The simplest: Minkovski metric $g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

rotational symmetry + translational symmetry.
The next simplest: rotational symmetry$\Rightarrow$ spherically symmetric S^2.

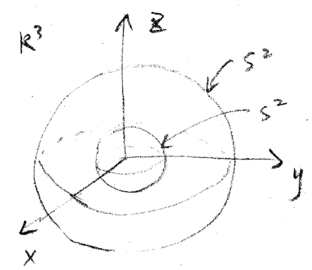
8.1.1 A handmoving argument

• Symmetry in differential geometry are described by Killing vectors.
• There exist a Frobenius's theorem which assert that: a spherically symmetric manifold can be foliated into spheres.
By foliation, we mean:

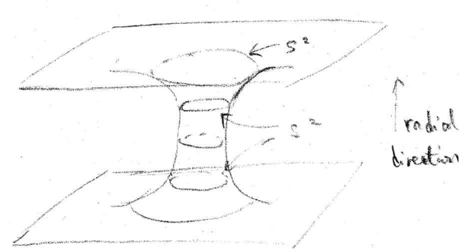
if we have an n-dimensional manifold foliated by m-dimensional submanifolds, we can use a set of m coordinate functions u^i on the m-dimensional submanifold to tell us where we are on this submanifold and use $n - m$ coordinate functions v^J to tell us which m-dimensional submanifold we are on.

1. Foliations by S^2:

   - R^3 Foliation by S^2 with an origin.



   - A different manifold with an $R \times S^2$ topology "Wormhole".

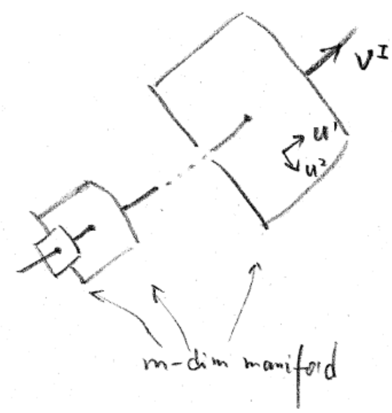


Here we suppressed one dimension of S^2 and use a circle to represent it. There is no point that looks like an "origin". [Imagine yourself being labeled on such an S^2: as you walk in radial direction, you experience a period of smaller radices and then it become large again.] All of these are possible because we do not have a flat spacetime like $\eta_{\mu\nu}$ any more. We are allowing any 4-d manifold to be a possible candidate of spacetime until we exclude them by experiments.

More precisely, an n-d manifold foliated by m-d submanifold has a metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{IJ}(v)dv^I dv^J + f(v)\gamma_{ij}(u)du^i du^j \tag{8.1}$$

Here v^I, $I = 1, \ldots, n-m$ are the coordinates that fixing which submanifold; u^i, $i = 1, \ldots, m$ are the coordinates on the m-dim submanifold. $\gamma_{ij}(u)$ is the metric of the submanifold.



Note:
(1) no cross terms $dv^I du^j$
(2) both $g_{IJ}(v)$ and $f(v)$ are functions of v^I alone.

8.1.2 The spherecally symmetric metric in $4-d$ spacetime

• Realize this theorem to our foliation by S^2, then our $m-dim$ submanifold is an $2-d$ sphere. Its metric can always be chosen as

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{8.2}$$

where (θ, ϕ) are the two coordinates (same angles as you learned in spherical cordinates in calculus).
Then the metric of our $4-d$ spacetime becomes from Eq(8.1)

$$ds^2 = g_{aa}(a,b)da^2 + g_{ab}(a,b)(dadb + dbda) + g_{bb}(a,b)db^2 + r^2(a,b)d\Omega^2 \tag{8.3}$$

I.e., we use a,b to denote the rest 2 coordinates, and $r^2(a,b)$ to denote the $f(v)$ function in Eq(8.1).
• We are free to do coordinate transform from (a,b) to (a,r), i.e., a reparameterization of b by r.
(Condition: assuming r depends on b; otherwise do $(a,b) \to (r,b)$; even otherwise, $\dfrac{\partial r}{\partial a} = 0 = \dfrac{\partial r}{\partial b}$.)
So the metric becomes $db(r) \to \dfrac{\partial b}{\partial r}dr$

$$\begin{aligned} ds^2 =& g_{aa}(a,b(r))da^2 + g_{ab}(a,b(r))\Big(da\frac{\partial b}{\partial r}dr + \frac{\partial b}{\partial r}drda\Big) \\ &+ g_{bb}(a,b)\Big(\frac{\partial b}{\partial r}\Big)^2 dr^2 + r^2 d\Omega^2 \end{aligned} \tag{8.4}$$

define new functions and rename

$$ds^2 = g_{aa}(a,r)da^2 + g_{ar}(a,r)(dadr + drda) + g_{rr}(a,r)dr^2 + r^2 d\Omega^2 \tag{8.5}$$

• Now we seek another transform of coordinates so that the crossing term $dadr + drda$ can be killed.
I.e., we seek a transform $(a, r) \to (t(a, r), s(a, r))$, s.t., $ds^2 = g_{tt}dt^2 + g_{ss}ds^2 + r^2(s, t)d\Omega^2$.
The transform $(a, r) \to (t(a, r), s(a, r))$ is the most general one we can do, however it spoiled the $r^2d\Omega^2$ term that we tried to put r as one of the coordinates. Then, let us try to see whether we can find a transform T

$$T : (a, r) \longrightarrow \Big(t(a, r), r\Big) \tag{8.6}$$

s.t. ①kill cross term,②keep r as a coordinate

$$ds^2 = g'_{tt}dt^2 + g'_{rr}dr^2 + r^2d\Omega^2 \tag{8.7}$$

Does this coordinate transform exist?
If yes, then since

$$\begin{aligned} ds^2 =& (g'_{tt}\frac{\partial^2 t}{\partial a^2})da^2 + g'_{tt}\frac{\partial t}{\partial a}\frac{\partial t}{\partial r}(drda + dadr) \\ &+ (\frac{\partial^2 t}{\partial r^2}g'_{tt} + g'_{rr})dr^2 + r^2d\Omega^2 \end{aligned} \tag{8.8}$$

Comparing to Eq(11.12), this require

$$g_{aa} = g'_{tt}\frac{\partial^2 t}{\partial a^2} \tag{8.9}$$

$$g_{ar} = g'_{tt}\frac{\partial t}{\partial a}\frac{\partial t}{\partial r} \tag{8.10}$$

$$g_{rr} = g'_{tt}\frac{\partial^2 t}{\partial r^2} + g'_{rr} \tag{8.11}$$

Three unknown functions $t(a, r)$, $g'_{tt}(t(a, r), r)$, $g'_{tt}(t(a, r), r)$ and three equations.
I.e., for any g_{aa}, g_{ar}, g_{rr}, we can find such $t(a, r)$, g'_{tt}, g'_{rr} that Eq (8.9)-Eq (8.11) are satisfied. $\Rightarrow$ The transform T always exists.
Using this transform (without having to know exactly its form), the metric with spherical symmetry can be written as (dropping ')

$$ds^2 = g_{tt}(t, r)dt^2 + g_{rr}(t, r)dr^2 + r^2d\Omega^2 \tag{8.12}$$

Claim: Any spherically symmetric metric in 4-d spacetime can be written into Eq (8.12) form.
• Signature of spacetime in G.R.(-+++)
Therefore one of $g_{tt}(t, r)$ and $g_{rr}(t, r)$ is negative and the other should be positive. We choose $sign(g_{tt}) = -1$, $sign(g_{rr}) = +1$ for now.
Then the metric sometimes is written as

$$ds^2 = -f(r, t)dt^2 + h(r, t)dr^2 + r^2d\Omega^2 \quad (f, h > 0) \tag{8.13}$$

or

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 \tag{8.14}$$

The metrics Eq (8.12), Eq (8.13), Eq (8.14) are the best we can do to a spherically symmetric spacetime.

8.1.3 Vacuum

The metric of a simplest spacetime (except Minkaski) is known now. One the other side of the E.E.

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{8.15}$$

the simplest form of $T_{\mu\nu}$ is vacuum, defined by $T_{\mu\nu} = 0$.
• In vacuum, the E.E. can be re-written

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{8.16}$$

Taking trace

$$R - \frac{1}{2}\cdot 4R = 8\pi T \quad [T \equiv T^{\mu}_{\mu}] \tag{8.17}$$

$$R = -8\pi T \tag{8.18}$$

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{8.19}$$

In vacuum, becomes

$$R_{\mu\nu} = 0 \tag{8.20}$$

• **Reminder:**
ⓐ $T_{\mu\nu}$ does not have to be zero everywhere in spacetime. As long as it is zero in a large chunk, we sometimes call it a "vacuum" solution.
ⓑ The word "vacuum" here is really a local word; meaning $T_{\mu\nu}$ only need to be zero locally.
ⓒE.E. is a local equation, like Maxwell's equations in its differential form. Everywhere $T_{\mu\nu} = 0$, the euqation $R_{\mu\nu} = 0$ has to be satisfied.
While for regions $T_{\mu\nu} \neq 0$, then $R_{\mu\nu}$ need to satisfy the $G_{\mu\nu} = 8T_{\mu\nu}$ at those region.

8.1.4 Solution of the vacuum E.E for a spherically symmetric spacetime

• Published January 13, 1916 by Karl Schwarzschild.
• Written during W.W.I when he is in the Russian front.
• Einstein's GR was known November 1915.
• First exact solution of E.E and reminds the most important one.
Einstein:

> I had not expected that one could formulate the exact solution of the problem in such a simple way.

• K.Schwarzschild died shortly after his paper was published in 1916, because of the disease he caught at Russian frontline. (May 11, 1916)
Start from

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2(d\theta^2 + sin^2 d\phi^2) \tag{8.21}$$

(t, r, θ, φ) being coordinates.
Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\mu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \tag{8.22}$$

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} (\Rightarrow \text{we need to only compute } \mu \leq \nu \text{ cases.}) \tag{8.23}$$

$$[g_{\mu\nu}] = \begin{bmatrix} -e^{2\alpha(t,r)} & & & \\ & e^{2\beta(t,r)} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{bmatrix} \tag{8.24}$$

$$[g^{\mu\nu}] = \begin{bmatrix} -e^{-2\alpha(t.r)} & & & \\ & e^{-2\beta(t,r)} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2 sin^2\theta} \end{bmatrix} \tag{8.25}$$

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2} g^{0\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}) \\ &= \frac{1}{2}(-e^{-2\alpha})(-e^{2\alpha})2 \cdot \partial_0\alpha \\ &= \partial_0\alpha \end{aligned} \tag{8.26}$$

$$\begin{aligned} \Gamma^0_{01} &= \frac{1}{2} g^{0\lambda}(\partial_0 g_{1\lambda} + \partial_1 g_{\lambda 0} - \partial_\lambda g_{01}) \\ &= \frac{1}{2}(-e^{-2\alpha})(-e^{2\alpha})2 \cdot \partial_1\alpha \\ &= \partial_1\alpha \end{aligned} \tag{8.27}$$

$$\begin{aligned} \Gamma^0_{11} &= \frac{1}{2} g^{0\lambda}(\partial_1 g_{1\lambda} + \partial_1 g_{\lambda 1} - \partial_\lambda g_{11}) \\ &= \frac{1}{2}(-e^{-2\alpha})(-e^{2\beta})\partial_0\beta \cdot 2 \\ &= e^{2(\beta-\alpha)}\partial_0\beta \end{aligned} \tag{8.28}$$

The rest

$$
\begin{aligned}
\Gamma^0_{\mu\nu} &= \frac{1}{2}g^{0\lambda}(\partial_\mu g_{\nu\mu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{2}g^{00}(\partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu}) \\
&= \frac{1}{2}g^{00}(\partial_\nu g^{00}) = 0 \\
&= \frac{1}{2}g^{00}(-\partial_0 g_{\nu\nu}) = 0 \quad (\nu \geq 2, \mu \leq \nu, \text{not summed}) && (8.29) \\
\Gamma^1_{\mu\nu} &= \frac{1}{2}g^{1\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{2}g^{11}(\partial_\mu g_{\nu 1} + \partial_\nu g_{1\mu} - \partial_1 g_{\mu\nu}) && (8.30) \\
\Gamma^1_{00} &= \frac{1}{2}g^{11}(-\partial_1 g_{00}) = \frac{1}{2}(e^{-2\beta})(+e^{+2\alpha})2\partial_1\alpha \\
&= e^{2(\alpha-\beta)}\partial_1\alpha && (8.31) \\
\Gamma^1_{01} &= \frac{1}{2}g^{11}(\partial_0 g_{11}) = \frac{1}{2}(e^{-2\beta})(e^{2\beta})2\partial_0\beta \\
&= \partial_0\beta && (8.32) \\
\Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11}) = \frac{1}{2}(e^{-2\beta})(e^{2\beta})2\partial_1\beta \\
&= \partial_1\beta && (8.33) \\
\Gamma^1_{\mu 2} &= \frac{1}{2}g^{11}(\partial_\mu g_{21} + \partial_2 g_{1\mu} - \partial_1 g_{\mu 2}) && (8.34) \\
\Gamma^1_{02} &= \Gamma^1_{12} = 0 && (8.35) \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(-\partial_1 g_{22}) = \frac{1}{2}(e^{-2\beta})(-2r) \\
&= -re^{-2\beta} && (8.36) \\
\Gamma^1_{\mu 3} &= \frac{1}{2}g^{11}(\partial_\mu g_{31} + \partial_3 g_{1\mu} - \partial_1 g_{\mu 3}) && (8.37) \\
\Gamma^1_{03} &= \Gamma^1_{13} = \Gamma^1_{23} = 0 && (8.38) \\
\Gamma^1_{33} &= \frac{1}{2}(e^{-2\beta})(-2r\sin^2\theta) \\
&= -rsin^2\theta e^{-2\beta} && (8.39) \\
\Gamma^2_{\mu\nu} &= \frac{1}{2}g^{2\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{2}g^{22}(\partial_\mu g_{\nu 2} + \partial_\nu g_{2\mu} - \partial_1 g_{\mu\nu}) && (8.40) \\
\Gamma^2_{\mu 0} &= \Gamma^2_{\mu 1} = 0 && (8.41) \\
\Gamma^2_{\mu 2} &= \frac{1}{2}g^{22}(\partial_\mu g_{22} + \partial_2 g_{2\mu} - \partial_2 g_{\mu 2}) = \frac{1}{2}g^{22}(\partial_\mu g_{\nu\nu}) && (8.42) \\
\Gamma^2_{02} &= 0 = \Gamma^2_{22} && (8.43) \\
\Gamma^2_{12} &= \frac{1}{2}g^{22}(\partial_1 g_{22}) \\
&= \frac{1}{2}\cdot\frac{1}{r^2}\cdot 2r = \frac{1}{r} && (8.44) \\
\Gamma^2_{\mu 3} &= \frac{1}{2}g^{22}(\partial_\mu g_{32} + \partial_3 g_{22} - \partial_2 g_{\mu 3}) && (8.45) \\
\Gamma^2_{03} &= \Gamma^2_{13} = \Gamma^2_{23} = 0 && (8.46) \\
\Gamma^2_{33} &= \frac{1}{2}\frac{1}{r^2}(-2r^2\sin\theta\cos\theta) = -\sin\theta\cos\theta && (8.47)
\end{aligned}
$$

$$\Gamma^2_{\mu\nu} = \frac{1}{2}g^{3\lambda}(\partial_\mu g_\nu\lambda + \partial_\lambda g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$
$$= \frac{1}{2}g^{33}(\partial_\mu g_{\nu 3} + \partial_\nu g_{3\mu} - \partial_3 g_{\mu\nu})$$
$$= \frac{1}{2}g^{33}(\partial_\mu g_{\nu 3} + \partial_\nu g_{3\mu} \tag{8.48}$$
$$\Gamma^3_{\mu 0} = 0 = \Gamma^3_{01} = \Gamma^3_{11} = \Gamma^3_{02} = \Gamma^3_{12} = \Gamma^3_{22} = \Gamma^3_{03} \tag{8.49}$$
$$\Gamma^3_{13} = \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r\sin^2\theta = \frac{1}{r} \tag{8.50}$$
$$\Gamma^3_{23} = \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r^2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta} \tag{8.51}$$
$$\Gamma^3_{33} = 0 \tag{8.52}$$
$$\tag{8.53}$$

Riemann tensor
......
Ricci tensor

$$(1)R_{00} = [\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta]$$
$$+ e^{2(\alpha-\beta)}[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\partial_1\beta - \frac{2}{r}\partial_1\alpha] \tag{8.54}$$
$$(2)R_{01} = \frac{2}{r}\partial_0\beta \tag{8.55}$$
$$(3)R_{11} = -[\partial_1^2\alpha + (\partial_1\alpha) - \partial_1\alpha\partial_1\beta - \frac{2}{r}\partial_1\beta]$$
$$+ e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] \tag{8.56}$$
$$(4)R_{22} = e^{-2\beta}[r(\partial_1\beta - \partial_1\alpha) - 1] + 1 \tag{8.57}$$
$$(5)R_{33} = R_{22}sin^2\theta \tag{8.58}$$

E.E. for vacuum: $R_{\mu\nu} = 0$

$$\partial_0\beta = 0 \tag{8.59}$$
$$\partial_0 R_{22} = e^{-2\beta}\cdot r\cdot(-\partial_0\partial_1\alpha) = 0 \tag{8.60}$$

These requires,

$$\beta = \beta(r) \tag{8.61}$$
$$\partial_0\alpha = h(r) \tag{8.62}$$
$$\alpha = \int h(r)dr + g(t) \tag{8.63}$$
$$\alpha = f(r) + g(t) \tag{8.64}$$

Aside:
The metric is then

$$ds^2 = -e^{2f(r)}e^{2g(t)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{8.65}$$

We can do another coordinate transform

$$dt \longrightarrow e^{-g(t)}dt \tag{8.66}$$

then the metric becomes

$$ds^2 = -e^{2f(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{8.67}$$

We call this a static metric–independent of x^0.
And the above procedure proved:
Any spherically symmetric vacuum metric can be transformed into a static metric.
Therefore sometimes in literature, a spherically symmetric & vacuum metric is often assumed to take the form

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + r^2 d\Omega^2 \tag{8.68}$$

Continue our solution to Eq (8.54) to Eq (8.58)

$$\begin{aligned} e^{2(\beta-\alpha)} \cdot (1) + (3) &= \frac{2}{r}(\partial_1\alpha + \partial_1\beta) = 0 \\ \partial_1\alpha + \partial_1\beta &= 0 \end{aligned} \tag{8.69}$$

$$(4) = e^{-2\beta}[r2\partial_1\beta - 1] + 1 = 0 \tag{8.70}$$

$$(1 - 2r\partial_1\beta)e^{-2\beta} = 1$$

$$\partial_1(re^{-2\beta}) = 1 \tag{8.71}$$

Solution:

$$e^{-2\beta} = 1 + \frac{\mu}{r} \quad \mu \text{ is arbitrary constant} \tag{8.72}$$

$$\alpha = -\beta(r) + h(t) \tag{8.73}$$

The metric becomes

$$\begin{aligned} ds^2 &= -e^{-2\beta(r)}e^{2h(t)}dt^2 + (1 + \frac{\mu}{r})^{-1} + r^2 d\Omega^2 \\ &= -e^{-2\beta(r)}dt^2 + (1 + \frac{\mu}{r})dr^2 + r^2 d\Omega^2 \end{aligned} \tag{8.74}$$

$$ds^2 = -\left(1 - \frac{\mu}{r}\right)dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{8.75}$$

This is the Schwarszchild solution of the vacuum Einstein equation for an spherically symmetric spacetimes.

• Fix the constant μ

The Eq(8.75) is asymptotically flat, i.e., $g_{\mu\nu} \xrightarrow{r\to\infty} \eta_{\mu\nu}$ Minkovski metric in spherical coordinates with r being the radius from origin. Therefore we interpret

Eq(8.75) as the exterior gravitational field of an isolated body. And we also call the "γ" is sm as the "radius" and the "t" as the "time".

μ can be fixed by comparing the Eq(8.75) with the metric of the weak field of a body with mass M.
In previous section, such a metric was obtained on Page 25.

$$\begin{cases} \bar{\gamma}_{00} = -4\phi \\ \bar{\gamma}_{ij} = \bar{\gamma}_{i0} = 0 \end{cases} \tag{8.76}$$

where $\phi = -\dfrac{GM}{r}$ is the gravitational potential.

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma} = \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(4\phi) \tag{8.77}$$

$$[\gamma_{\mu\nu}] = \begin{pmatrix} -4\phi + 2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{pmatrix} \gamma_{\mu\nu} \qquad = -2\phi\delta_{\mu\nu} \tag{8.78}$$

then we have

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{8.79}$$

In spherical coordinates

$$g_{00} = -(1 + 2\phi) \tag{8.80}$$

$$g_{\gamma\gamma} = (1 - 2\phi) \tag{8.81}$$

Comparing with Eq(8.75), we get $\mu = -2GM$.
Then finally we arrive at the celebrated Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega \tag{8.82}$$

Comments:
(a) Birkhoff's theorem: S.M. is the unique spherically symmetric vacuum solution. By "unique", all other solutions can be reached from the SM by coordinate transformation.
(b) The source dose not need to be static. E.g. a collapsing star, a supernova explosion. As long as the process is spherically symmetric.
(c) This last point (b) is like in e&m, where a radial redistribution of total charge dose not affect the electric field outside. In particular, little radiation is generated. Similarly, since the gravitational field is not much changed during redistribution little gravitational wave will be generated.

8.1.5 Singularity in the SM

Singularity? What is it.

Mathematically, a (set of) points at which some given mathematical object becomes not defined or not "well-behaved". E.g., infinite or not-differentiable.

Complex analysis: $\frac{1}{x^n} \to$ order n singularity;

$$f(z) = \frac{sin(z)}{z} \qquad z = 0 \quad \text{removable singularity} \tag{8.83}$$

1. Here,

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{8.84}$$

Two superfacial singularities of the spacetime

$$\gamma = 0 \tag{8.85}$$

$$\gamma = 2M \tag{8.86}$$

Since the metric becomes poorly behaved. But is this a precise enough reason to say they are singularities?
Answer: No. True physical singularities of spacetime should be defined from the behavior of some physical process, not simply from the metric.

In particular, metric $g_{\mu\nu}$'s components depend on the coordinate system you choose for the manifold.
E.g., in polar coordinates, the metric of $\mathbb{R}^2$

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{8.87}$$

At point $r = 0$, $g_{rr} = 0$; but we know that point is perfectly ok. So, we need to study coordinate independent quantities to characterize singularities.
Here is our definition:

Definition 8.1.1. *Points at which any scalar constructed from the metric becomes infinite.*
- *By "scalar", $(0, 0)$ type tensor.*
- *Singularities defined this way are sometimes called "curvature singularities".*
- *Many times we also associate a physical condition to the definition: These points have to be reachable by travelling a finite distance along a geodesic.*
- *Other definitions exist.*

Scalars we usually consider

$$R_{\alpha\beta\sigma\rho}g^{\alpha\sigma}g^{\beta\rho} = R \quad \text{Ricci scalar} \tag{8.88}$$

$$R_{\alpha\beta\sigma\rho}R_{\mu\nu\lambda\eta}g^{\alpha\mu}g^{\beta\nu}g^{\sigma\lambda}g^{\rho\eta} = R^{\mu\nu\lambda\eta}R_{\mu\nu\lambda\eta} \tag{8.89}$$

$$R_{\alpha\beta\sigma\rho}R_{\mu\nu\lambda\eta}g^{\alpha\sigma}g^{\nu\lambda}g^{\beta\nu}g^{\rho\eta} = R^{\mu\nu}R_{\mu\nu} \tag{8.90}$$

$$R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu} \tag{8.91}$$

$$\vdots \tag{8.92}$$

$$\vdots \tag{8.93}$$

$$\text{even high orders in } R_{\mu\nu\sigma\rho}. \tag{8.94}$$

Non-singularity?
• Check all scalars at these points.
• Geodesics behave well at these points.
2. Analysis of $r = 0$ and $r = 2M$

$$R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} = \frac{48G^2M^2}{r^6} \tag{8.95}$$

Then $r = 0$ is a singularity point, since the above blows up. While the $r = 2M$ manifold is not singular from the point of view of $R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma}$. Indeed, you can check other curvature scalars, at $r = 2M$, they are not singular.

Proof. Find a coordinate system near $r = 2M$, s.t. the metric is well-defined and well-behaved.
Consider the coordinate transform

$$(t, r) \longrightarrow (v, r) \tag{8.96}$$

where

$$v = t + r + 2Mln|\frac{r}{2M} - 1| \tag{8.97}$$

The metric then will take the form

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2d\Omega^2 \tag{8.98}$$

This metric behaves completely fine at $r = 2M$.

□

• There exist (many) other coordinate transforms that will make the metric well behaved at $r = 2M$.

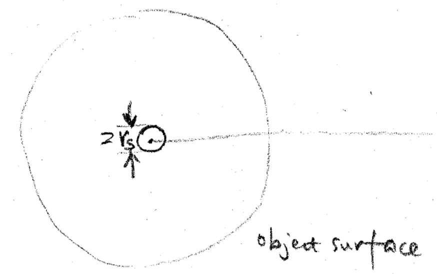
• Though the $r = 2M$ surface is not singular, we will show later on that it is an very very interesting surface, which has astonishing properties.

8.1.6 Interior solution

The manifold of $r = 2M$ happens for an object with mass M at

$$r_s \equiv 2M = \frac{2GM}{c^2} \approx 2.95(\frac{M}{M_\odot})km \quad \text{Schwarzchild radius} \tag{8.99}$$

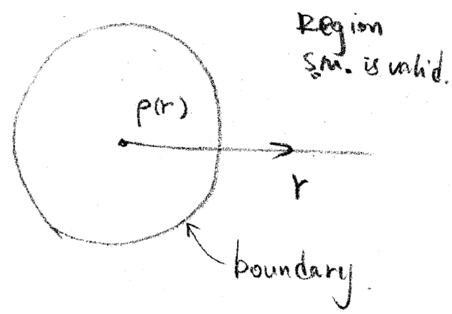
For our sun, $r_s = 2.95km \ll R_{sun}$; earth, $r_s = 2.95 \times 3.0 \times 10^{-6}km = 9mm \ll R_{earth}$.



Therefore for most objects, the Schwarzschild radius is well inside the object, where the S.M. solution is not valid anyway.

E.E with spherically symmetric fluid

We wish to :
① Study the gravitational field inside an spherical object.
② Compare with Newtonian limit.
③ See how to join with the exterior Schwarzschild vacuum solution.



• Suppose that the equation of state of the interior matter is $p = p(\rho)$.
• The interior matter density $\rho = \rho(r)$.
• Interior reached equilibrium & therefore static.
We use a metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{8.100}$$

E.E. becomes

$$(a) \quad G_{00} = \frac{1}{r^2}e^{2\alpha}\frac{d}{dr}[r(1-e^{-2\beta})] \tag{8.101}$$

$$(b) \quad G_{rr} = -\frac{1}{r^2}e^{-2\beta}(1-e^{-2\beta}) + \frac{2}{r}\frac{d\alpha}{dr} \tag{8.102}$$

$$(c) \quad G_{\theta\theta} = r^2 e^{-2\beta}\Big[\frac{d^2\alpha}{dr^2} + (\frac{d\alpha}{dr})^2 + \frac{1}{r}\frac{d\alpha}{dr} - \frac{d\alpha}{dr}\cdot\frac{d\beta}{dr} - \frac{d\beta}{dr}\cdot\frac{1}{r}\Big] \tag{8.103}$$

$$(d) \quad G_{\phi\phi} = sin^2\theta G_{\theta\theta} \tag{8.104}$$

Assuming the interior is a perfect fluid, then

$$T^{\mu\nu} = (\rho + p)v^{\mu}v^{\nu} + pg^{\mu\nu} \tag{8.105}$$

Here v^{μ} is the for-velocity field of the fluid satisfying $v^{\mu}v_{\mu} = -1$.
Since the star is static, v^{μ} only has t component, so

$$v^0 v^0 g_{00} = -1 \tag{8.106}$$
$$v^0 = e^{-2\alpha(r)} \tag{8.107}$$
$$v_0 = -e^{\alpha(r)} \tag{8.108}$$
$$v^i = 0 \tag{8.109}$$

These produce a $T^{\mu\nu}$:

$$(1) \quad T_{00} = \rho e^{2\alpha} \tag{8.110}$$
$$(2) \quad T_{rr} = \rho e^{2\beta} \tag{8.111}$$
$$(3) \quad T_{\theta\theta} = r^2 p \tag{8.112}$$
$$(4) \quad T_{\phi\phi} = sin^2\theta T_{\theta\theta} \tag{8.113}$$

And they have to satisfy the conservation law

$$\nabla_{\mu} T^{\mu\nu} \tag{8.114}$$

Computing the Chiristofel symbols, then the covariant derivitive, we get the only non-vanishing component $\nu = \gamma$,

$$(5) \quad (\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr} \tag{8.115}$$

Now, we have all components collected in our hands, we need to solve them.

$$\begin{cases} (a) = (1) \cdot 8\pi \\ (b) = (2) \cdot 8\pi \\ (c) = (3) \cdot 8\pi \\ (d) = (4) \cdot 8\pi \\ (5) \end{cases}$$

we have 4 functions $\alpha(r)$, $\beta(r)$, $\rho(r)$, $p(r)$.
We need to know $\alpha(r)$ and $\beta(r)$. To do this, perhaps we need to know as an input:

$\rightarrow$ both $\rho(r)$ and $p(r)$; or
$\rightarrow$ only one of $\rho(r)$ and $p(r)$ and the other can be fixed; or
$\rightarrow$ perhaps both $\rho(r)$ and $p(r)$ have to have a fixed form to let $\alpha(r)$ and $\beta(r)$ have a solution.

Which is the case, let us try to solve the system.
First let us trade-off $\beta(r)$ by $m(r)$

$$m(r) \equiv \frac{1}{2}r(1 - e^{-2\beta(r)}) \tag{8.116}$$

Then the $(0,0)$ component of the E.E becomes

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{8.117}$$

while $(1,1)$ component becomes

$$\frac{d\alpha(r)}{dr} = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 3m(r))} \tag{8.118}$$

using this in the conservation Eq(8.115), we get

$$\frac{dp}{dr} = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r - 2m)} \tag{8.119}$$

The $(2,2)$ component equation, which is the longest, can be checked to be automatically satisfied using Eq(8.117), Eq(8.118), Eq(8.119) and definition of $m(r)$.
So effectively, four functions $m(r)$, $\alpha(r)$, $p(r)$, $\rho(r)$ and three equations. Only when one of them is fixed, the other three can be solved.
Eq(8.117) has solution of $m(r)$ in $\rho(r)$

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx + a \tag{8.120}$$

This allow us to express the g_{rr} component

$$g_{rr} = e^{2\beta} = 1 - \frac{2m(r)}{r} = 1 - \frac{2\int_0^r 4\pi x^2 \rho(x) dx + 2a}{r} \tag{8.121}$$

$$g_{rr}(r \to 0) \longrightarrow 1 - \frac{2a}{r} \tag{8.122}$$

We want a smooth metric at $r = 0$ and therefore $a = 0$. I.e.

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx \tag{8.123}$$

• Then assuming $m(r)$ is known, the next simplest equation is Eq(8.119).
After solving $p(r)$ from Eq(8.119) and putting into Eq(8.118), we can solve $\alpha(r)$.
• So, all of $m(r)$, $p(r)$ and $\alpha(r)$ can be fixed up to some integral constants by knowing $\rho(r)$.
• Unfortunately, we can not solve Eq(8.119) generally without knowing some particular form of $\rho(r)$.

Therefore we study some simple $\rho(r)$ next.

Before doing that, some comments: (1) The formula of $m(r)$ is the same as the mass formula in Newtonian gravity. However, it is not exact the mass in G.R., because in G.R., the proper volume element is $\sqrt{|g|}dx^3 = e^{-2\beta(r)}r^2 \sin\theta dr d\theta d\phi$. Mass is

$$M(x) = \int_0^r 4\pi x^2 \rho(x)\Big[1 - \frac{2m(x)}{x}\Big]^{-\frac{1}{2}} dx \tag{8.124}$$

$$M(x) > m(r) \quad \text{since} \quad 2m(x) \leq x$$

The difference $\Delta M \equiv M(x) - m(x)$ is the gravitational binding energy.
(2) In Newtonian limit, $\rho \gg p$, $m(r) \sim \rho(r)R^3 \gg pr^3$, $m(r) \ll r$, then Eq (8.118) becomes

$$\frac{d\alpha(r)}{dr} \approx \frac{m(r)}{r} \tag{8.125}$$

which is the Poisson's equation for Newtonian gravity potential.
(3) Eq (8.119) is called Tolman-Oppenheimer-Volkoff equation. It states a relation between p and ρ that a equilibrium and static fluid should satisfy in a spherical gravity field.
In Newtonian limit, it becomes

$$\frac{dp}{dr} = -\frac{\rho m(r)}{r^2} \tag{8.126}$$

which is Newtonian hydrostatic equilibrium equation.

Pressure, density & stability of stars

(1) Now we have to specify a form of $\rho(r)$.
We consider an incompressible fluid of density ρ.

$$\rho(r) = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

Then immediately from Eq (8.117)

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r \leq R \\ \frac{4}{3}\pi R^3 \rho_0 \equiv M & r > R \end{cases} \tag{8.127}$$

(2) Then from Eq (8.119), now $\rho(r)$ is solvable

$$p(r) = \begin{cases} \rho_0 \Big[\dfrac{(1-\frac{2M}{R})^{\frac{1}{2}} - (1-\frac{2M}{R}\cdot\frac{r^2}{R^2})^{\frac{1}{2}}}{(1-\frac{2M}{R}\cdot\frac{r^2}{R^2})^{\frac{1}{2}} - 3(1-\frac{2M}{R})^{\frac{1}{2}}}\Big] & r \leq R \\ 0 & r > R \end{cases} \tag{8.128}$$

There was an integral constant that we used to set the boundary condition $p(R) = 0$.
We would like $p(r)$ to be bounded at all $r \geq 0$:

$$|p(r)| < \infty \quad \text{for} \quad r \geq 0 \tag{8.129}$$

The extreme of $p(r)$ occur at points

$$(1-\frac{2M}{R}\cdot\frac{r^2}{R^2})^{\frac{1}{2}}-3(1-\frac{2M}{R})^{\frac{1}{2}}=0 \tag{8.130}$$

$$\text{or at } r=0,\ r=\infty \tag{8.131}$$

$$\text{or at } \frac{dp(r)}{dr}=0 \tag{8.132}$$

We can solve these potentially extreme points.
Eq (8.130)⇒

$$r^2=R^2(q-\frac{4R}{M}) \tag{8.133}$$

Eq (8.132)⇒

$$r=0 \tag{8.134}$$

$$\text{boundaries } r=0 \quad r=\infty \text{ (gone)} \tag{8.135}$$

① When $q-\frac{4R}{M}<0$, Eq (8.133) has no real solution.
Extreme happens at the center of the star

$$p(r=0)=\rho_0\Big[\frac{1-(1-\frac{2M}{R})^{\frac{1}{2}}}{3(1-\frac{2M}{R})^{\frac{1}{2}}-1}\Big] \tag{8.136}$$

$$q-\frac{4R}{M}<0\Rightarrow 3(1-\frac{2M}{R})^{\frac{1}{2}}-1>0 \tag{8.137}$$

Therefore, $p(r)$ is bounded. ② When $q-\frac{4R}{M}\geq 0$, Eq (8.133) has real positive solution

$$\begin{aligned} r &= R(q-\frac{4R}{M})^{\frac{1}{2}} \\ &\leq R \quad \text{since } r\geq 2m(r), \\ R &\geq 2M \text{ for } r=\textit{constant} \text{ surface to be static and spacelike} \end{aligned}$$

I.e., $p(r=R(q-\frac{4R}{M})^{\frac{1}{2}})$ will blow up. Summarizing ① and ②, we get ① has to be the case for a stable star.
That is

$$M<\frac{4R}{M} \tag{8.138}$$

or,

$$M<\frac{4}{9}\cdot\frac{1}{(3\pi\rho)^{\frac{1}{2}}} \qquad \text{since} \quad \frac{4}{3}\pi R^3\rho=M \tag{8.139}$$

Physical statement: A object with low density but large mass can not be stable.
E.g., A gas planent can not be too heavy if it's stable.
Comments:

① Is this statement due to G.R.? What about Newtonian case?
In Newtonian limit, Poisson's equation(Eq (8.126)) lets us solve

$$p(r) = \begin{cases} \frac{2}{3}\pi\rho_0^2(R^2 - r^2) & r \le R \\ 0 & r > R \end{cases}$$

This is everywhere finite.
Therefore no stability statement can be said about the star.
② What if $\rho \neq$ constant? Results: 1) For $\frac{d\rho}{dr} \le 0$, and the star has a fixed radius R, one can always prove the maximum stable mass is

$$M_{max} = \frac{4}{9}R \tag{8.140}$$

2) For $\frac{d\rho}{dr} \le 0$, $|p(r)| \le \rho_0$ and $\frac{dp(\rho)}{d\rho}$ at small ρ, there always exist an upper limit of mass for stable star.
• Along this line, many interesting and grand conclusions can be drawn about stars.
Consult some modern astronomy books.
3) Eq (8.118) for $\rho =$ constant case can be solved analytically. The result is quite long so we will not write it out.
But beyond $r = R$, we can verify that the solution becomes

$$ds^2 = -(1 - \frac{2M}{r}dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \qquad r \ge R \tag{8.141}$$

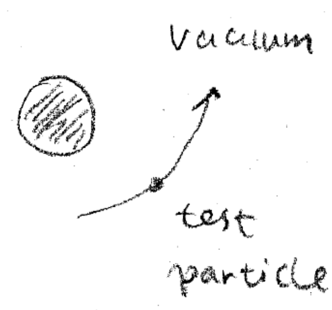
I.e., the two part of the metric join smoothly.

8.2 Geodesics in the Schwarzschild Spacetime

The interest here is to study how the geodesics behave in the vacuum part of the Schwarzschild in spacetime.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{8.142}$$

The reason to study geodesics is apparent: all matter, including satellite/moons around planets, planets around stars or space travellers, follow their geodesics. We would like to know what happens to them.



8.2.1 The general geodesic equation and its simplification

Geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{8.143}$$

where λ is a affine parameter.
For timelike geodesics, λ is chosen as propertime.
For spacelike geodesics, it can be chosen as proper space.
For null geodesics, it is some affine parameter.

Also, we have the normalization condition

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu} = -\kappa \tag{8.144}$$

here, $\kappa = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$

The Christoffel symbols (non-zero):

$$\Gamma^0_{01} = \frac{M}{r(r-2M)} \tag{8.145}$$

$$\Gamma^1_{00} = \frac{M}{r^3}(r-2M) \tag{8.146}$$

$$\Gamma^1_{11} = -\frac{M}{r(r-2M)} \tag{8.147}$$

$$\Gamma^1_{22} = -(r-2M)\sin^2\theta \tag{8.148}$$

$$\Gamma^2_{12} = \frac{1}{r} \tag{8.149}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \tag{8.150}$$

$$\Gamma^3_{13} = \frac{1}{r} \tag{8.151}$$

$$\Gamma^3_{23} = \frac{\cos\theta}{\sin\theta} \tag{8.152}$$

• 4-components of the geodesic equation:

$$\frac{d^2t}{d\lambda^2} + \frac{2M}{r(r-2M)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{8.153}$$

$$\frac{d^2r}{d\lambda^2} + \frac{M}{r^3}(r-2M)\Big(\frac{dt}{d\lambda}\Big)^2 - \frac{M}{r(r-2M)}\Big(\frac{dr}{d\lambda}\Big)^2 - (r-2M)\Big[\Big(\frac{d\theta}{d\lambda}\Big)^2 + \sin^2\theta\Big(\frac{d\phi}{d\lambda}\Big)^2\Big] = 0 \tag{8.154}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\Big(\frac{d\phi}{d\lambda}\Big)^2 = 0 \tag{8.155}$$

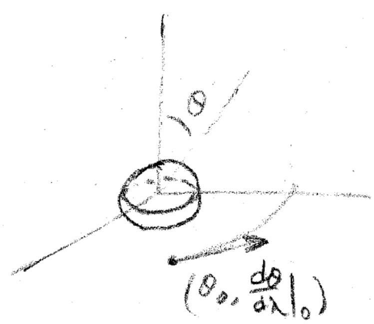
$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0 \tag{8.156}$$

Normalization condition

$$-\Big(1-\frac{2M}{r}\Big)\Big(\frac{dt}{d\lambda}\Big)^2+\Big(1-\frac{2M}{r}\Big)^{-1}\Big(\frac{dr}{d\lambda}\Big)^2+r^2\Big(\frac{d\phi}{d\lambda}\Big)^2+r^2\sin^2\theta\Big(\frac{d\theta}{d\lambda}\Big)^2=-\kappa \tag{8.157}$$

where $\kappa=\begin{cases}+1 & \text{timelike}\\ 0 & \text{null}\\ -1 & \text{spacelike}\end{cases}$

Simplification:
Now this is a system of ode's.
• Solution will depend on initial position and 1-st derivative. Notice $\theta\to\pi-\theta$ is a symmetry of the system (and the metric), then if initial position and tangent vector lies in the plane $\theta=\frac{\pi}{2}$, then the test particle will remain in this plane, because it should not deviate from this plane to any direction.



On the other hand, any initial position and tangent vector can be brought to this plane through rotation of coordinate system. Therefore, whitout losing any generality, we can set

$$\theta(\theta)=\frac{\pi}{2}\quad\text{for all }\lambda \tag{8.158}$$

And this apparent solves Eq (8.155).
• Then using this in Eq(8.156) $\cdot\, r^2$, we realize its a total derivative

$$\frac{d\Big(r^2\dfrac{d\phi}{d\lambda}\Big)}{d\lambda}=0 \tag{8.159}$$

i.e.,

$$r^2\frac{d\phi}{d\lambda}=L \tag{8.160}$$

• The Eq (8.153) can also be tackled, treating $r(\lambda)$ and $\dfrac{dr}{d\lambda}$, as known functions, multiplying by $\Big(1 - \frac{2M}{r})\Big)$. we see

$$\text{Eq(8.153)} \Rightarrow \frac{d}{d\lambda}\Big[\Big(1 - \frac{2M}{r}\Big)\frac{dt}{d\lambda}\Big] = 0 \tag{8.161}$$

$$\Big(1 - \frac{2M}{r}\Big)\frac{dt}{d\lambda} = E \tag{8.162}$$

So, once $r(\lambda)$ is known, using Eq (8.160), Eq (8.162), we can get $t(\lambda)$ and $\varphi(\lambda)$.
• The Eq (8.157), after multiplying $\Big(1 - \frac{2M}{r}\Big)$, and using Eq (8.160) & Eq (8.162), becomes,

$$-E^2 + \Big(\frac{dr}{d\lambda}\Big)^2 + \Big(1 - \frac{2M}{r}\Big)\Big(\frac{L^2}{r^2} + \kappa\Big) = 0 \tag{8.163}$$

This can also be written as

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + V(r) = \frac{1}{2}E^2 \tag{8.164}$$

where,

$$\begin{aligned} V(r) &= \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)\kappa + \frac{L^2}{2r^2}\Big(1 - \frac{2M}{r}\Big) \\ &= \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)\Big(\kappa + \frac{L^2}{r^2}\Big) \end{aligned} \tag{8.165}$$

• Finally the longest equation Eq (8.154), after substituting $\frac{d\varphi}{dr}$, $\frac{dt}{dr}$, becomes

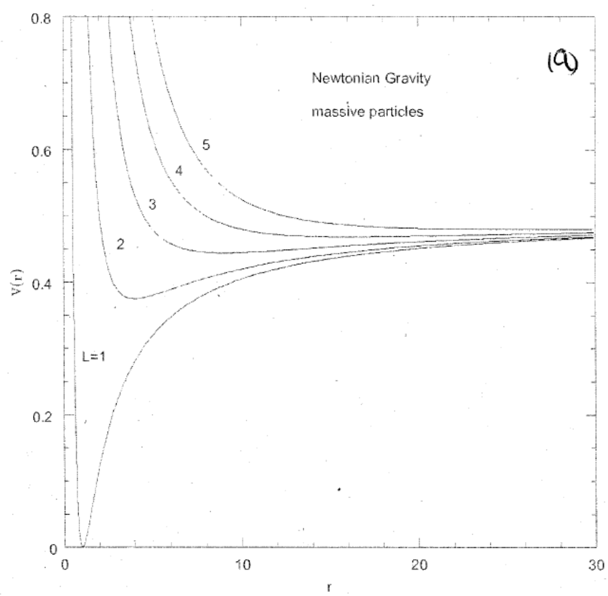
$$\frac{d}{d\lambda}\Big[\Big(\frac{d\lambda}{d\lambda}\Big)\Big/\Big(1 - \frac{2M}{r}\Big) + \frac{L^2}{r^2} - \frac{E^2}{1 - \frac{2M}{r}}\Big] = 0 \tag{8.166}$$

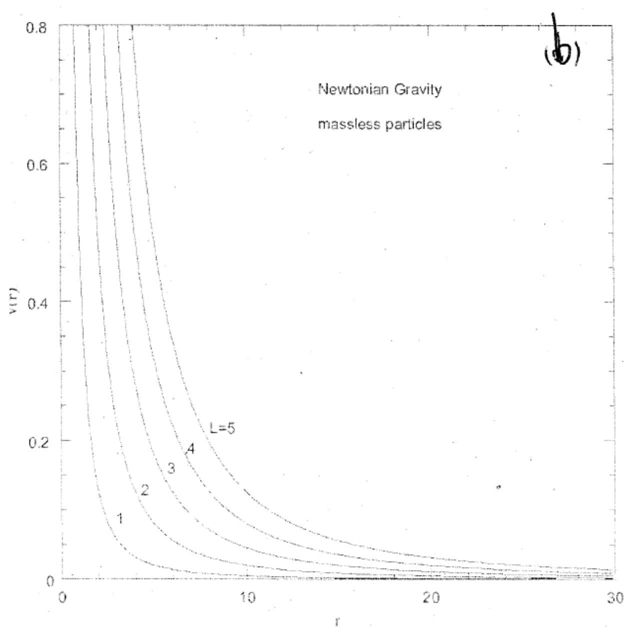
From this we get

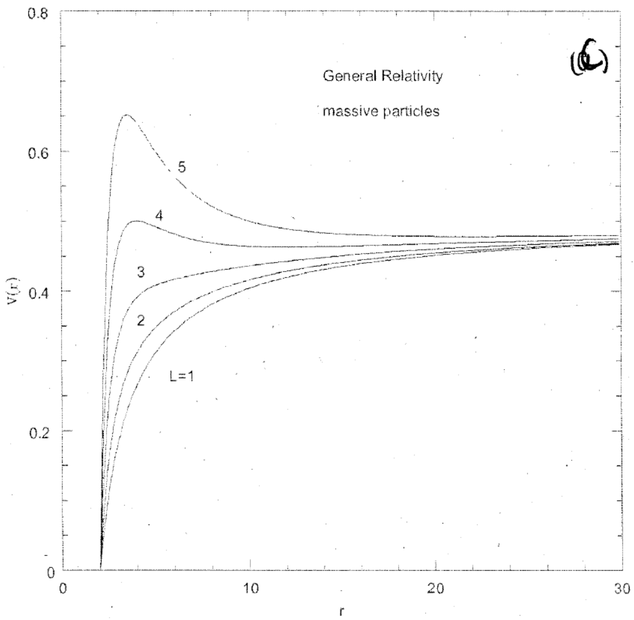
$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + V'(r) = \frac{1}{2}E^2 \tag{8.167}$$

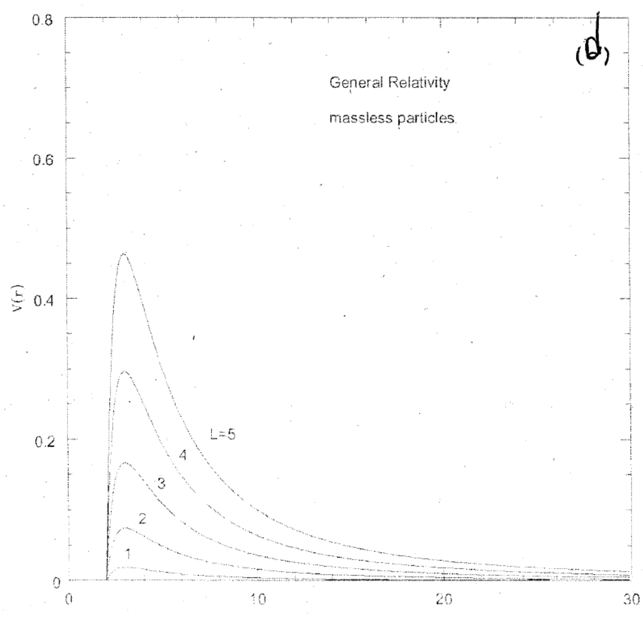
where $V'(r) = \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)\Big(\frac{L^2}{r^2} + c\Big)$ for any constant c.
Then comparing with Eq (8.164), we see that $c = \kappa$ in order to have solutions.
Summary:
$\Rightarrow$ We finally have 3 equations, Eq (8.162), Eq (8.160) and Eq (8.164) with $r(\lambda)$, $\varphi(\lambda)$ and $t(\lambda)$ undetermined.
$\Rightarrow$ Once $r(\lambda)$ is determined, the rest can be fixed easily.
$\Rightarrow$ The Eq (8.164) equation is the same as a unit mass particle of energy $\frac{E^2}{2}$ moving in 1-dimensional effective potential $V(r)$.
$\Rightarrow$ The integral constants L and E corresponds to the test particle's angular momentum and energy per unit mass respectively. They are conserved because they are constants of the motion.

$\Rightarrow$ In Newtonian gravity, the equation of motion would be the same except the potential $V_{\text{Newton}} = \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)\kappa + \frac{1}{2}\frac{L^2}{r^2}$.
I.e., without the $-\frac{ML^2}{r^3}$ term.






(c)
General Relativity
massive particles
5
4
3
2
L=1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r


(d)
General Relativity
massless particles
L=5
4
3
2
1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r

Chapter 9

Analysis of the motions

We compare the motion of massive and massless test particles in G.R. and Newtonian gravity, hoping to find some detectable effects that can confirm the validity of G.R.
First let us point out some general conclusions that are not specific to G.R. or Newtonian gravity, but to any central force potential

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + V(r) = \text{constant} \tag{9.1}$$

(a) If $V(r)$ allows a stable circular motion for a test particle ($\frac{dr}{d\lambda} = 0$), their $V(r)$ should have local minima. I.e., $\frac{dV(r)}{dr} = 0$ has solutions, and $\frac{d^2V(r)}{dr^2} > 0$ at that r.
In our case,

$$\frac{dV}{dr} = \frac{\kappa M r^2 - L^2 r + 3ML^2\delta}{r^4} = 0 \tag{9.2}$$

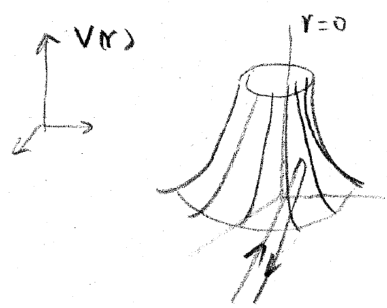
where $\kappa = \begin{cases} 1 & \text{massive} \\ 0 & \text{massless} \end{cases}$ and $\delta = \begin{cases} 1 & \text{G.R.} \\ 0 & \text{Newtonian} \end{cases}$.
The solution then is

$$r_c = \begin{cases} 3M\delta & \kappa = 0, \delta = 1 \\ \dfrac{L^2 \pm \sqrt{L^4 - 12M^2L^2\delta}}{2M} & \kappa = 1 \end{cases}$$

They are potential locations of stable circular motion.
(b) If the potential has a local minimum, then there can exist bound orbits. The bound orbits which are not circular will oscillate around radius of stable circular motion.

9.1 Newtonian potential, massive particle

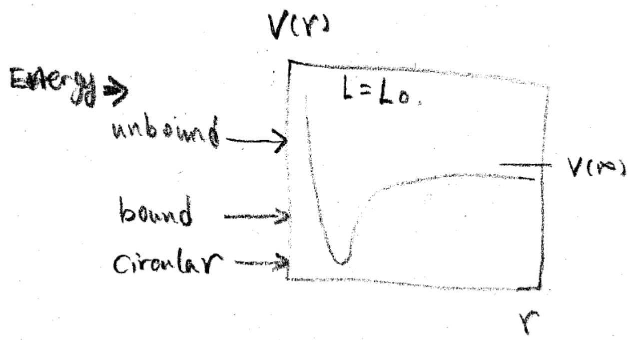
• $\kappa = 0$, $\delta = 0$.
There is not a solution for r_c, no circular motion, no bound orbits.
• Straight line in $\mathbb{R}^3$ since in Newtonian gravity, gravitational force is zero because of zero mass.
• Smaller L means closer initial shooting direction.



• Imagine a massless particle hitting a potential vocano like this.
⋆ Slowing down as approaching, then bounce back, all along straight lines.
⋆ If you have higher energy, you can psss straightly.
⋆ If you have closer aiming direction, you have an lower version of the potential, i.e., a lower mountain, and you can come closer to $r = 0$.

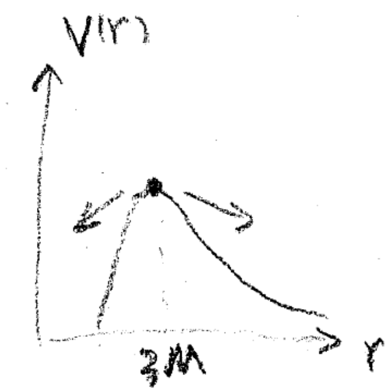
9.2 Newtonian potential, massive particle

• $\kappa = 1$, $\delta = 0$, there exist a local minimum.
Stable circular orbit at $r_c = \dfrac{L^2}{M}$.
Bound orbit around this radius.
• If your energy $E > V(r = \infty)$, you have unbound orbit. (From classical mechanics, bound orbits: ellipse; unbound orbits: para-bola or hyper-bola.)



G.R. geodesic motions
The difference in potentials is the $\sim \frac{1}{r^3}$ term, therefore it will be manifest when r is small.

9.3 G.R. massless particle

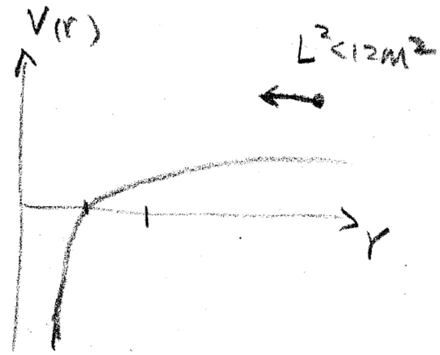


• $\kappa = 0$, $\delta = 1$.
$r_c = 3M$ is an maximum.
No stable bound orbit: either fly in or out.

9.4 G.R. massive particle

• $\kappa = 1$, $\delta = 1$.

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M} \tag{9.3}$$



• If $L^2 < 12M^2$, r_c is not real $V(r)$ has no minimum or maximum. (It do has a $V(r) = 0$ point at $r = 2M$.) From the Eq (8.164), indeed one can show that for particles with $\frac{dr(\text{initial})}{d\lambda} \leq 0$ and $L^2 < 12M^2$, the particle will fall directly the $r = 2M$ surface and enter it.

• If $L^2 > 12M^2$, then $r_{c-} \equiv \frac{L^2 - \sqrt{L^4 - 12L^2M^2}}{2M}$ is a maximum while $r_{c+} \equiv \frac{L^2 + \sqrt{L^4 - 12L^2M^2}}{2M}$ is a local minimum. So stable circular motion is possible at r_{c+}.

Bound orbit exist if $\frac{1}{2}E^2 < V(\infty) = \frac{1}{2}$.

Unbound orbit exist if $V(\infty) < \frac{1}{2}E^2 < V(r-)$.

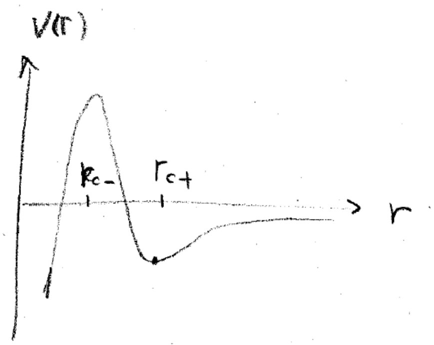
• Because $L^2 > 12M^2$, $r_{c+} > 6M$, $3M < r_{c-} < 6M$, where both $6M$ are obtained when $L^2 = 12M^2$ and the $3M$ for r_{c-} is when $L^2 \to \infty$.

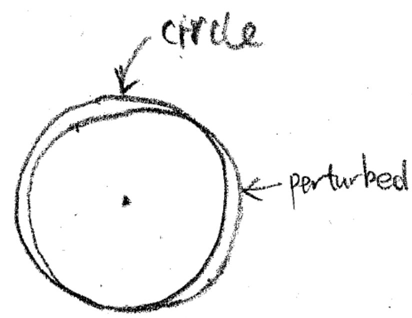
Noting in Newtonian case, $r_c = \frac{L^2}{M} = r_{c+}$ $(L^2 \to \infty)$, therefore the $L \to \infty$ limit in G.R. corresponds to the Newtonian limit.

Chapter 10

Observational effects of G.R.

10.1 Precession of perihelia





• The radial geodesic was

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + V(r) = \frac{1}{2}E^2 \tag{10.1}$$

We showed that when $L^2 > 12M^2$, there exist a r_{c+} for massive particle. If $\dfrac{1}{2}E^2 = V(r_{c+})$, then the test particle will do circular motion.
• When $\frac{1}{2}E^2$ is slightly larger than $V(r_{c+})$, the $r(\lambda)$ will oscillate around r_{c+}. Using,

$$r = r_{c+} + \delta \sin(\omega\lambda + c) \tag{10.2}$$

$$\frac{1}{2}E^2 = V(r_{c+} + \delta) \tag{10.3}$$

We solve the frequency of oscillation to be

$$\omega_r^2 = \frac{d^2V}{dr^2}\Big|_{r=r_{c+}} = \frac{M(r_{c+}-6M)}{r_{c+}^3(r_{c+}-3M)} \tag{10.4}$$

where we used r_{c+} to substitute L^2.
• The above was all about radial geodesics.
We had two simple equation Eq (8.162) and Eq (8.160) that we need to solve too.
From Eq (8.160), when $r \approx r_{c+}$,

$$\omega_\phi^2 = \left(\frac{d\phi}{d\lambda}\right)^2 = \left(\frac{L}{r^2}\right)^2 = \left(\frac{L}{r_{c+}^2}\right)^2 = \mathcal{O}(\delta) \tag{10.5}$$

$$\omega_\phi^2 = \frac{M}{r_{c+}^2(r_{c+}-3M)} \tag{10.6}$$

(a) In the limit of Newtonian gravity

$$L \to \infty$$

then,

$$r_{c+} \gg M$$

Apparently, $\omega_r^2 \approx \omega_\phi^2$. Both r and ϕ return to their initial values simultaneously. Orbit is closed. (Indeed in N.G., $\frac{\omega_r}{\omega_\phi} = \frac{n}{m}$, for n, m integers.)
(b) In G.R., $\omega_r \neq \omega_\varphi$ cause a precession of angle at which a minimum r is achieved.
These minimum r points are called “perihelia”.
The precession due to G.R. is

$$\begin{aligned}\omega_p &\equiv \omega_\phi - \omega_r \\ &= \frac{M}{r_{c+}^2(r_{c+}-3M)} - \frac{M(r_{c+}-6M)}{r_{c+}^3(r_{c+}-3M)} \qquad (10.7)\\ &= \frac{3M^{\frac{3}{2}}}{r_{c+}^{\frac{5}{2}}} \qquad (10.8)\end{aligned}$$

It was proven that for elliptical orbit with semimajor axis a and eccentricity e, the precession to lowest order is

$$\omega_p \approx \frac{3M^{\frac{3}{2}}}{(1-e^2)a^{\frac{5}{2}}} \tag{10.9}$$

★ This result was applied to the precession of Mecury around the Sun. Einstein at 1916 showed that the G.R. explained it well. And this success was an important factor for the early acceptance of G.R. by people.

10.2 Bending of light ray

In last section, we used the Eq (8.160) for timelike geodesics to solve the frequency of angular motion. We now look at the Eq (8.160) for null geodesics.
• The Eq (8.164) allows us to solve

$$\dot{r} = \Big[E^2 - \frac{L^2}{r^2}\Big(1 - \frac{2M}{r}\Big)\Big]^{\frac{1}{2}} \tag{10.10}$$

while Eq (8.160) can be written as

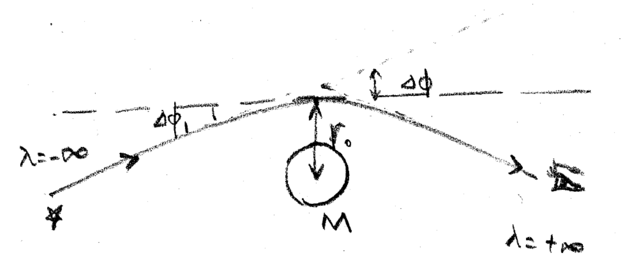
$$\dot{\phi} = \frac{L}{r^2} \tag{10.11}$$

combining

$$\frac{d\phi}{dr} = \frac{L}{r^2}\Big[E^2 - \frac{L^2}{r^2}\Big(1 - \frac{2M}{r}\Big)\Big]^{\frac{1}{2}} \tag{10.12}$$

We would like to consider

$$\Delta\phi \equiv \phi(\lambda = +\infty) - \phi(\lambda = -\infty) \tag{10.13}$$



Graphically:
At $\lambda = \pm\infty$, the trajectories are almost straight lines. Because at $r \to \infty$, the metric is Newtonian-like.
• For the distance $r(\lambda)$, there is a turning point where r stops decreasing and starts increasing. That is when $\dfrac{dr}{d\lambda} = 0$.
From Eq (8.164), that is when

$$V(r) = \frac{1}{2}E^2 \tag{10.14}$$

$$\frac{L^2}{2r^2} - \frac{ML^2}{r^3} = \frac{1}{2}E^2 \tag{10.15}$$

$$r^3 - \Big(\frac{L}{E}\Big)^2 (r - 2M) = 0 \tag{10.16}$$

The solution, denoted by r_0, is

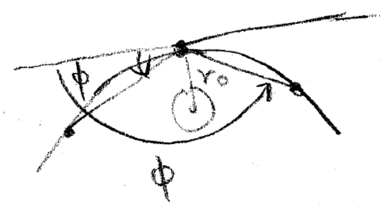
$$r_0 = \frac{2b}{\sqrt{3}}\cos\Big[\frac{1}{3}\cos^{-1}\Big(-\frac{3^{\frac{3}{2}}M}{b}\Big)\Big] \tag{10.17}$$

$b \equiv \dfrac{L}{E}$ is called "apparent impact parameter".

• Then the angle of the tangent vector accumulated from $r = \infty$ to $r = r_0$ is given using Eq (10.12)

$$\begin{aligned}\Delta\phi_1 &= \int_{\infty}^{r_0} \frac{L}{r^2}\left[E^2 - \frac{L^2}{r^3}(r-2M)\right]^{-\frac{1}{2}} dr \\ &= \int_{\infty}^{r_0} \frac{dr}{\left[r^4 b^{-2} - r(r-2M)\right]^{\frac{1}{2}}}\end{aligned} \tag{10.18}$$

While the whole bending angle, by symmetry (or by defining ϕ = angle from baseline, and then $\pi - \phi$ on the other half), is two time $\Delta\phi_1$.



After changing of variable $u = \dfrac{1}{r}$,

$$\Delta\phi = |2\Delta\phi_1| = 2\int_0^{\frac{1}{r_0}} \frac{du}{(b^{-2} - u^2 + 2Mu^3)^{\frac{1}{2}}} \tag{10.19}$$

This expends on two parameters b and M; note $r_0 = r_0(b, M)$.

• This is an integral quantity, the Newtonian limit is in $M = 0$.

$$\Delta\phi(M \to 0) = 2\int_0^{\frac{1}{b}} \frac{du}{(b^{-2} - u^2)} = \pi \qquad r_0(M \to 0) = b \tag{10.20}$$

Completely in agreement that null geodesics go straight in Newtonian gravity.

• In G.R., we would like to see the bending of light passing points with some known distance from the center, not light with particular apparent impact factor.

Then we replace b^2 in Eq (10.19) by $b^2 = \frac{r_0^3}{r_0 - 2M}$ from Eq (10.17)

$$\Delta\phi = 2\int_0^{\frac{1}{r_0}} \frac{du}{(r_0^{-2} - 2Mr_0^{-3} - u^2 + 2Mu^3)^{\frac{1}{2}}} \tag{10.21}$$

$$\Big\downarrow \text{small } M \text{ limit}$$

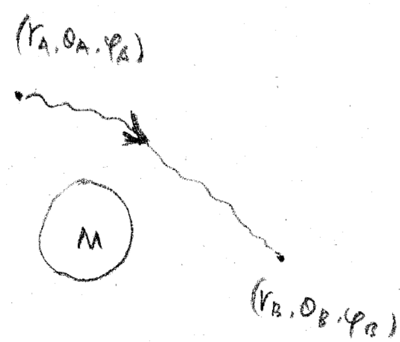
$$\begin{aligned}&= \Delta\phi(M=0) + \frac{\partial(\Delta\phi)}{\partial M}\Big|_{M=0} \cdot M + \mathcal{O}(M^2) \\ &= \pi + \frac{4}{b}\end{aligned} \tag{10.22}$$

• 1919 Eddington, Arthur measured this bending and find good agreement with G.R. First major prediction confirmed for G.R.

10.3 Gravitational Redshift

We have been working with the radial and angular geodesics equations. Now we also use the t-component of the geodesics to derive another effect of G.R.

• Consider a light ray was emitted at time $t'(\lambda = \lambda_i)$ from point $(r_A, \theta_A, \varphi_A)$ and was received at time $t'(\lambda = \lambda_f)$ by someone at point $(r_B, \theta_B, \varphi_B)$.



So its null geodesics is described by

$$[x^\mu] = [t'(\lambda), r'(\lambda), \theta'(\lambda), \varphi'(\lambda)] \tag{10.23}$$

$$\begin{cases} r'(\lambda_i) = r_A, \quad r'(\lambda_f) = r_B \\ \theta'(\lambda_i) = \theta_A, \quad \theta'(\lambda_f) = \theta_B \\ \varphi'(\lambda_i) = \varphi_A, \quad \varphi'(\lambda_f) = \varphi_B \end{cases} \tag{10.24}$$

Consider another light ray, emitted at some later coordinate time

$$t' = t(\lambda) + \Delta t \quad (\Delta t \text{ is } \lambda \text{ independent constant}) \tag{10.25}$$

with same initial r, θ, φ and four velocity.
Then from Eq (8.153), Eq (8.154), Eq (8.155), Eq (8.156), and Eq (8.157), it is not hard to see they are also satisfied by

$$[x'^\mu] = [t'(\lambda) + \Delta t, r'(\lambda), \theta'(\lambda), \varphi'(\lambda)] \tag{10.26}$$

I.e., the above is the geodesic solution of light ray. Then at coordinate time $t'(\lambda_f) + \Delta t$, the second ray will reach $(r(\lambda_f), \theta(\lambda_f), \varphi(\lambda_f))$ which is $(r_B, \theta_B, \varphi_B)$. That is, the light ray emitted Δt coordinate time after, from the same source, will be receive Δt coordinate time later by the receiver at the same position.

• The source fixed at $(r_A, \theta_A, \varphi_A)$ and receiver fixed at $(r_B, \theta_B, \varphi_B)$ usually are massive materials. Therefore their world lines are timelike.
Their tangent vectors should satisfy the normalization condition

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1 \tag{10.27}$$

Denoting the source's propertime as λ_A, and that of the receiver as λ_B. Their 4-components as $[x^\mu_A] = (t_A, r_A, \theta_A, \varphi_A)$ and $[x^\mu_B] = (t_B, r_B, \theta_B, \varphi_B)$. Then

$$-\left(1 - \frac{2M}{r_A}\right)\left(\frac{dt_A}{d\lambda_A}\right)^2 = -1 \qquad (r_A, \theta_A, \varphi_A) \text{ are fixed.} \tag{10.28}$$

$$-\left(1 - \frac{2M}{r_B}\right)\left(\frac{dt_B}{d\lambda_B}\right)^2 = -1 \qquad (r_B, \theta_B, \varphi_B) \text{ are fixed.} \tag{10.29}$$

I.e., the propertime and coordinate time has relation for A:

$$d\lambda_A = \left(1 - \frac{2M}{r_A}\right)^{\frac{1}{2}} dt_A \tag{10.30}$$

for B:

$$d\lambda_B = \left(1 - \frac{2M}{r_B}\right)^{\frac{1}{2}} dt_B \tag{10.31}$$

Integrating both sides, noting r_A, r_B does not depend on t_A, t_B

$$\Delta\lambda_A = \left(1 - \frac{2M}{r_A}\right)^{\frac{1}{2}} \Delta t_A \tag{10.32}$$

$$\Delta\lambda_B = \left(1 - \frac{2M}{r_B}\right)^{\frac{1}{2}} \Delta t_B \tag{10.33}$$

Now if two light ray was emitted by a time difference Δt_A and received by a coordinate time difference Δt_B, we showed $\Delta t_A = \Delta t_B$. Then,

$$\frac{\Delta\lambda_A}{\Delta\lambda_B} = \left(\frac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}}\right)^{\frac{1}{2}} \tag{10.34}$$

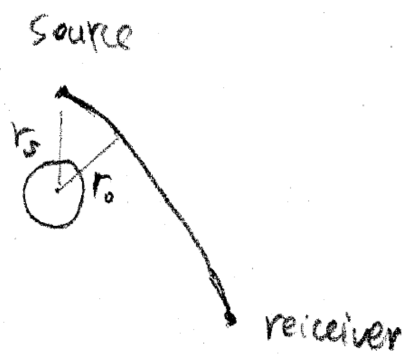
In terms of frequency, the emitting and receiving frequency would be

$$\frac{\omega_A}{\omega_B} = \frac{\Delta\lambda_B}{\Delta\lambda_A} = \left(\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}}\right)^{\frac{1}{2}} \tag{10.35}$$

For $r_B > r_A > 2M$ (the usual case: r_A is the surface of a star, r_B is the earth-star distance), $\omega_B < \omega_A$, or $\lambda_B > \lambda_A$ due to constancy of light speed.
$\Rightarrow$ A resshift.
• First conclusive verification: 1959 the Pownd-Rebka experiment.

10.4 Time delay of light ray

Now we try to use the Eq (8.162) equation of null geodesic to show another measurable effect of G.R.– Time delay of light signal.



• The Eq (8.162) equation again

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \tag{10.36}$$

and the Eq (8.164)

$$\frac{dr}{d\lambda} = \left[E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}\right]^{\frac{1}{2}} \tag{10.37}$$

Then, we get

$$\frac{dt}{dr} = \frac{\frac{dt}{d\lambda}}{\frac{dr}{d\lambda}} = \left(1 - \frac{2M}{r}\right)\left[1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-\frac{1}{2}} \tag{10.38}$$

where $b = \dfrac{L}{E}$. Then the coordinate time taken for the light from a source at r_s to its bending point at r_0 is

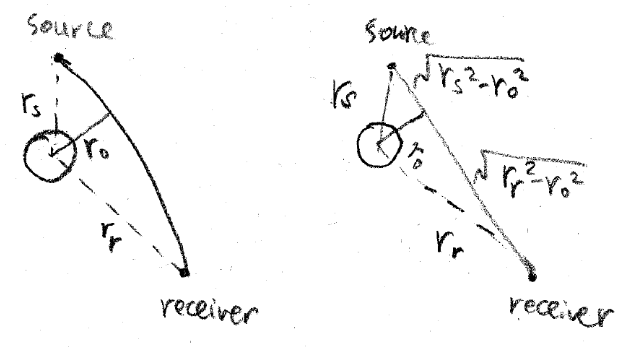
$$\Delta t\Big|_{r_0}^{r_s} = \int_{r_0}^{r_s} \left(1 - \frac{2M}{r}\right)^{-1}\left[1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-\frac{1}{2}} dr \tag{10.39}$$

• The result depends on b, which is not easy to measure, therefore we, as before, us $b = r_0\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$ to replace it. We also work on the first order in M, the result is

$$\Delta t\Big|_{r_0}^{r_s} = \sqrt{r_s^2 - r_0^2} + 2M \ln\left(\frac{r_s + \sqrt{r_s^2 - r_0^2}}{r_0}\right) + M\left(\frac{r_s - r_0}{r_s + r_0}\right)^{\frac{1}{2}} \tag{10.40}$$

The time from sending until receiving by an receiver at r_r, is then

$$\begin{aligned} \Delta t =& \Delta t\Big|_{r_0}^{r_s} + \Delta t\Big|_{r_0}^{r_r} \qquad (10.41) \\ =& \sqrt{r_s^2 - r_0^2} + \sqrt{r_r^2 - r_0^2} + 2M\left[\ln\left(\frac{r_s + \sqrt{r_s^2 - r_0^2}}{r_0}\right) + \ln\left(\frac{r_r + \sqrt{r_r^2 - r_0^2}}{r_0}\right)\right] \\ &+ 2M\left[\left(\frac{r_s - r_0}{r_s + r_0}\right)^{\frac{1}{2}} + \left(\frac{r_r - r_0}{r_r - r_0}\right)^{\frac{1}{2}}\right] \end{aligned}$$



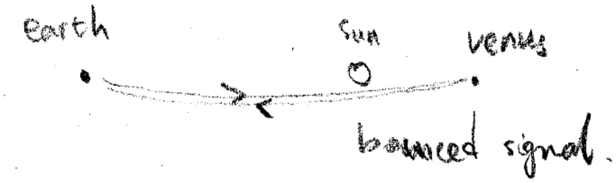
• The propertime at the receiver radius r_r is related to the propertime by

$$\Delta\lambda = \left(1-\frac{2M}{r_r}\right)^{\frac{1}{2}}\Delta t \tag{10.42}$$

$$\downarrow \text{First order of } M$$

$$= \Delta t - \frac{M}{r}\left(\sqrt{r_s^2-r_0^2}+\sqrt{r_r^2-r_0^2}\right)$$

⇒ The $\sqrt{r_s^2-r_0^2}+\sqrt{r_r^2-r_0^2}$ is the Newtonian limit result.
⇒ The rest terms are G.R. corrections.
⇒ This effect is first-noticed by Irwin.I.Shapiro. First confirmed in 1966.



Chapter 11

Cosmology

11.1 Homogeneity and Isotropy

Definition 11.1.1. *A spacetime is said to be homogenous if there exists a one-parameter family of spacelike hyper surfaces Σ_t foliating the spacetime; and for each t and any points $p, q \in \Sigma_t$, there exists an distance-preserving map which takes p into q.*

About the observational support of the homogeneity of universe, there are some data but not very conclusive.

Definition 11.1.1. *A spacetime is said to be isotropic around a point p, if for two unit spatial tangent vectors in the tangent space of p: $v_1 \in T_p$, $v_2 \in T_p$, there exist an distance-preserving map that can take v_1 to v_2.*

Observational support of isotropy around earth:
Cosmic microwave background first observed 1964-1965. Current CMB temperature $T = 2.72548 \pm 0.00057$ Kelvin.
• The homogeneity and isotropy implies that the spacetime of the universe can be foliated into $R \times \Sigma_t$. Its metric takes the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(u)du^i du^j \tag{11.1}$$

(1) The "$-dt^2$" term might contain an coefficient function of t; but we can always scale t to make it 1.
(2) The $a(t)$ is called scale factor; γ_{ij} is the metric on Σ_t, $i, j = 1, 2, 3$.
(3) The coordinates that make the metric free of $dtdu^i$ term and spacelike part $du^i du^j$ only depend on a single function of $a(t)$, is called "comoving coordinates".
(4) An observer has constant u^i is called a "comoving observer".

11.2 Robertson-Walker metric

• Now let us see what else requirement the homogeneity and isotropy can put on the metric.
Look at ${}^{(3)}R_{ij}{}^{kl}$ (Riemann tensor calculated from the metric γ_{ij} of the 3-dimensional surface) at any point p.
For a tensor in the tensor space of rank $(0,2)$, T_{kl}, we have at point p,

$$({}^{(3)}R_{ij}{}^{kl}T_{kl} = T'_{ij} \tag{11.2}$$

is another element in the $(0,2)$ rank tensor space. So $({}^{(3)}R_{ij})^{kl}$ can be thought as an linear map from

$$\text{rank }(0,2)\text{ tensor space} \longrightarrow \text{rank }(0,2)\text{ tensor space}$$

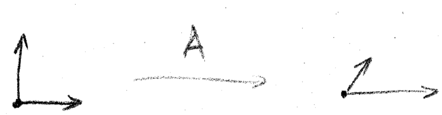
In linear algebra, you learned

$$A^{ab}T_a = T'_b \tag{11.3}$$

A^{ab} is a linear transform. It is eigendecomposible

$$A = Q\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1} \tag{11.4}$$

If the eigenvalues of A are different, then there exists set of vectors, whose norm will not be scaled uniformly.



Therefore if we want the scaling of norms all vectors to be the same, then the eigenvalues of A has to be the same.
So for $({}^{(3)}R_{ij})^{kl}$, due to isotropy we want the scaling of norms of all tensors at p to be the same, therefore the eigenvalues of $({}^{(3)}R_{ij})^{kl}$ should be equal.
I.e.,

$$\begin{aligned}
({}^{(3)}R_{ij})^{kl} =& Qk\delta_i^k\delta_j^l Q^{-1} + Qk'\delta_j^k\delta_i^l Q^{-1} \\
& \Big\downarrow ({}^{(3)}R_{ij})^{kl} = -({}^{(3)}R_{ji})^{kl} \\
=& Qk\delta_{[i}^k\delta_{j]}^l Q^{-1} \\
& \Big\downarrow \text{transfromation of coordinates} \\
({}^{(3)}R_{ij})^{kl} =& k\delta_{[i}^k\delta_{j]}^l
\end{aligned}$$

• Lowering the indices by γ_{ij}, we get

$$^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{11.5}$$

The Ricci tensor becomes

$$^{(3)}R_{jl} = 2k\gamma_{jl} \tag{11.6}$$

The metric on the hypersurface should also be spherically symmetric (isotropy+homogeneity). Then we can propose the following form (similar to the derivation of Schwarzschild metric)

$$d\sigma_\Sigma^2 = \gamma_{ij}du^i du^j = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta)d\varphi^2 \tag{11.7}$$

I.e., $[\gamma_{ij}] = \begin{bmatrix} e^{2\beta(r)} & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{bmatrix}$.

The Ricci tensor of this metric is

$$^{(3)}R_{11} = \frac{2}{r}\partial_1\beta \tag{11.8}$$

$$^{(3)}R_{22} = e^{-2\beta}(r\partial_1\beta - 1) + 1 \tag{11.9}$$

$$^{(3)}R_{33} = {}^{(3)}R_{22}\sin^2\theta \tag{11.10}$$

Matching with Eq (11.6), we get

$$\beta(r) = -\frac{1}{2}\ln(1 - kr^2), \quad e^{2\beta} = \frac{1}{1-kr^2} \tag{11.11}$$

Then

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big] \tag{11.12}$$

Another change of coordinates

$$r \to \sqrt{|k|}r$$
$$a \to \frac{a}{\sqrt{|k|}}$$
$$k \to \frac{k}{|k|}$$

leaves Eq (11.12) invariant. Therefore what matters are the signs of k.

• The Robertson-Walker metric

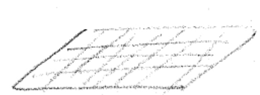
$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big] \tag{11.13}$$

$k = -1$: constant negative curvature on Σ. The metric/spacetime/universe is called "open".
$k = 0$: no curvature on Σ. The metric/spacetime/universe is called "flat".
$k = 1$: constant positive curvature on Σ. The metric/spacetime/universe is

called "closed".
1. For $k = 0$, the metric on Σ can be computed to

$$d\sigma_\Sigma^2 = dx^2 + dy^2 + dz^2 \tag{11.14}$$

after usual spherical coordinates to Euclidean coordinates transform. Therefore, locally the metric describes an flat $\mathbb{R}^3$.



2. For $k = 1$, we can transform $r = \sin\chi$, then

$$d\sigma_\Sigma^2 = d\chi^2 + \sin^2\chi d\Omega^2 \tag{11.15}$$

Globally, the metric describes a 3-sphere. This is not a 2-sphere you can draw, which is embedded in $\mathbb{R}^3$. It does not have boundary but its size is finite.
3. For $k = -1$, transform $r = \sinh\psi$ brings metric to

$$d\sigma_\Sigma^2 = d\psi^2 + \sinh^2\psi d\Omega^2 \tag{11.16}$$

Space with constant negative curvature.
2-D section e.g.

Therefore sometimes, the RW metric is also written as

$$ds^2 = -dt^2 + a(t)\begin{cases} d\chi^2 + \sin^2\chi d\Omega^2 \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi d\Omega^2 \end{cases} \tag{11.17}$$

• Based only on homogeneity and isotropy, we fixed the metric from 10 unknown functions of all spacetime coordinates, to 3 discrete cases with 1 function $a(t)$ of a single coordinates. (Power of symmetry)

11.3 Classical cosmological models

11.4 Friedmann equations

The E.E

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{11.18}$$

work out $G_{\mu\nu}$.
From the RW metric Eq (11.13), the Ricci tensors

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{11.19}$$
$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \tag{11.20}$$
$$R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \tag{11.21}$$
$$R_{33} = R_{22}\sin^2\theta \tag{11.22}$$

where $\dot{a} = \dfrac{da}{dt}$, $\ddot{a} = \dfrac{d^2a}{dt^2}$.
Ricci scalar

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \tag{11.23}$$
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{11.24}$$

• For the energy-momentum tensor, we assume that the universe is filled with perfect fluid:

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu} \tag{11.25}$$

where U_μ is the four-velocity of the fluid.
In a comoving frame, the fluid will be at rest

$$U_\mu = (1, 0, 0, 0)$$

And so

$$[T_{\mu\nu}] = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 1 & & & \end{bmatrix} \tag{11.26}$$
$$T = T^\mu_\mu = -\rho + 3p \tag{11.27}$$

• The E.E produce $(0,0)$ component

$$-3\frac{\ddot{a}}{a} = 4\pi(\rho + 3p) \tag{11.28}$$

$(1,1)$, $(2,2)$, $(3,3)$ components give only 1 equation

$$\frac{\ddot{a}}{a} + 2\Big(\frac{\dot{a}}{a}\Big)^2 + 2\frac{k}{a^2} = 4\pi(\rho - p) \tag{11.29}$$

Use Eq (11.28) in Eq (11.29), we finally get two simplified equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \tag{11.30}$$
$$\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{11.31}$$

(1) They are called "Friedmann equations". RW metrics that obey these equations describes an "FRW" universe.
(2) We also define a parameter

$$H \equiv \frac{\dot{a}}{a} \tag{11.32}$$

Hubble parameter: measure how fast the universe expands at a particular time. Current measured value $H > 0 \Rightarrow$ universe expanding.

11.5 Different fluids

We finally have 3 Friedmann equations Eq (11.30) and Eq (11.31), and three unknowns $a(t)$, $\rho(t)$ and $p(t)$. We therefore need a equation of state $p = p(\rho)$. Essentially, the perfect fluid relavant for cosmology obey the simple EOS.

$$p = \omega\rho \tag{11.33}$$

where ω is a constant.
Then we can start to solve them.

$$\frac{d(\dot{a}^2)}{dt} = d\Big(\frac{8\pi}{3}\rho a^2)/dt \tag{11.34}$$

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}(\dot{\rho}a^2 + 2\rho a\dot{a}) \tag{11.35}$$

Using Eq (11.30), Eq (11.33)

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \tag{11.36}$$

solving this we obtain

$$\rho = \rho_0 a^{-3(1+\omega)} \tag{11.37}$$

ρ_0 is the density when $a(t) = 1$.
To further solve $a(t)$ and p, we need to specify particular value for ω.

11.5.1 Dust universe

Collisionless, nonrelativistic matter. Therefore no pressure

$$p = 0 \quad \omega = 0 \tag{11.38}$$

• Immediately, $\rho = \rho_0 a^{-3}$.
This is easy to understand: *the space volume increase as a^3 while the total number of the dust is conserved, then the number density decrease like a^{-3}, and the energy density $\rho \sim n \cdot m_{each} \cdot c^2 \sim a^{-3}$.*
• The Eq (11.31) now becomes

$$\dot{a}^2 - \frac{8\pi}{3}\rho_0\frac{1}{a} + k = 0, \quad \frac{8\pi}{3}\rho_0 \equiv c \tag{11.39}$$

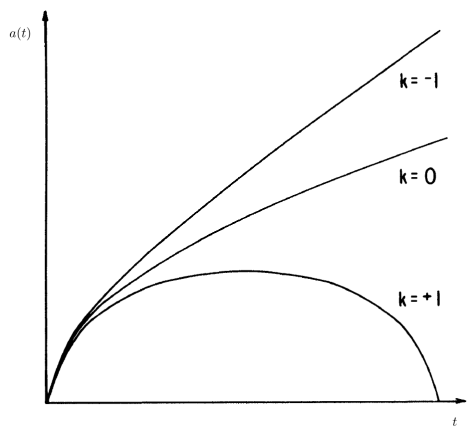
This equation looks quite simple but the result can only be expressed in the form of a parameter equation.

$k = +1$: $a = \frac{1}{2}c(1 - \cos\eta)$, where η is related to t by: $t - \frac{1}{2}c(\eta - \sin\eta)$.

$k = 0$: $a = \left(\frac{9c}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}}$.

$k = -1$: $a = \frac{1}{2}c(\cosh\eta - 1)$, $t = \frac{1}{2}c(\sinh\eta - \eta)$.

Eq (11.30) is automatically satisfied.



(A) Since $\dot{a}(\text{now}) > 0$, we are expanding. Left side of the figure.
(B) Depending on k, the universe might
expand forever: $k1$, $k = 0$,
shrink eventually: $k = 1$.
(C) In anycase, in the past $a \to 0$.
• The $a \to 0^+$ state dose not mean the size of the universe is zero.
• Rather, the distance between matter is approaching zero.
⇒ Everything is compressed together but the universe is still expected to be infinite.

11.5.2 Radiation solution

For radiation $p = \frac{1}{3}\rho$, $\omega = \frac{1}{3}$.
• Then immediately from Eq (11.37)

$$\rho = \rho_0 a^{-4} \tag{11.40}$$

This is not hard to understand either:
The number density goes $\rho \sim a^{-3}$, and then the wave length goes like $\lambda \sim a$,

$\omega \sim a^{-1}$, then the energy density become

$$\rho \sim a^{-4} \tag{11.41}$$

• The Eq (11.31) becomes after the same procedure as in the case of dust

$$\dot{a} - \frac{8\pi}{3}\rho_0 \frac{1}{a^2} + k = 0, \quad c \equiv \frac{8\pi}{3}\rho_0 \tag{11.42}$$

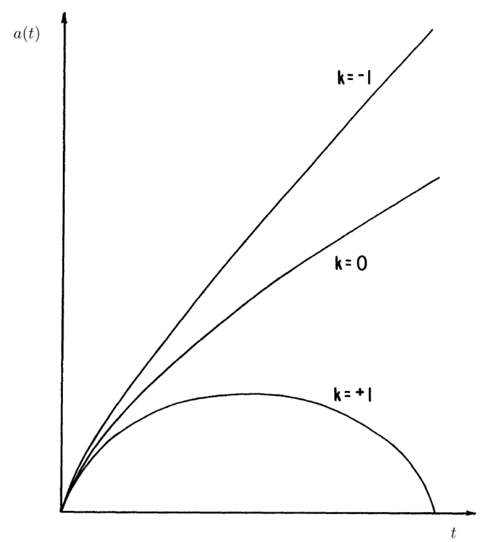
The solution is then given by

$$k = +1 \quad a = \sqrt{c}\Big[1 - \Big(1 - \frac{t}{\sqrt{c}}\Big)^2\Big]^{\frac{1}{2}} \tag{11.43}$$

$$k = 0 \qquad a = (4c)^{\frac{1}{4}} t^{\frac{1}{2}} \tag{11.44}$$

$$k = -1 \quad a = \sqrt{c}\Big[\Big(1 + \frac{t}{\sqrt{c}}\Big)^2 - 1\Big]^{\frac{1}{2}} \tag{11.45}$$

Eq (11.30) is automatically satisfied.



(A) Previous statements still hold.
(B) Most importantly, there was also a point of time that $a(t) \to 0^+$.

11.5.3 Vacuum energy included

It is also possible that there is an vacuum energy with

$$T^{\text{vac}}_{\mu\nu} = -\frac{\Lambda}{8\pi} g_{\mu\nu} \tag{11.46}$$

or,

$$\rho = -p = \frac{\Lambda}{8\pi} \tag{11.47}$$

$\omega = -1$ in EOS. $\rho = \omega = rho$.
The E.E for $T_{\mu\nu}$ including such an vacuum energy should also be modified to

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{11.48}$$

here $T_{\mu\nu} = T_{\mu\nu}(\text{other}) + T_{\mu\nu}^{\text{vac}}$.

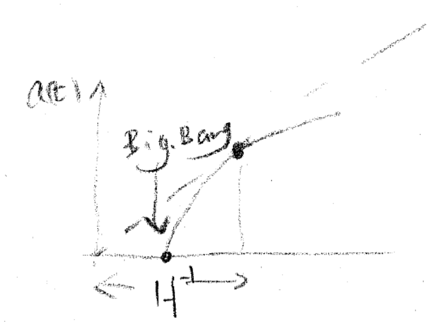
However, we will not further assume other forms of energy-momentum tensor or persuit fuither solutions. –Beyond the scope of this course.

11.6 Big bang

The perfect fluids studied in 2 are crude simplified versions. Reality differs.
• Observationally $H = \frac{\dot{a}}{a} \approx 40 - 90km/sec/M_{pc} > 0$ ($M_{pc} = 3 \times 10^{24}cm$)
• Without assuming a particular form of EOS, as long as $\rho > 0$, $p \geq 0$, from Eq (11.30), we see that

$$\ddot{a} = -\frac{4\pi}{3}(\rho + 3p)a < 0 \tag{11.49}$$

• Then $\dot{a}(t = \text{past}) > \dot{a}(t = \text{now}) = 40 \sim 90km/sec/M_{pc} \cdot a(\text{now}) > 0$.
I.e., the universe as accelerating faster.
If it was expanding at a constant value $\dot{a}_{\text{now}}$ then some time $T = \frac{a_{\text{now}}}{\dot{a}_{\text{now}}}$ ago, $a(t)$ would be zero.
• This means the universe started from a density infinite and Ricci curvature $R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)$ infinite singularity. This singularity is called Big Bang.



(1) Consequently, the age of universe $A_{\text{universe}} < \frac{a}{\dot{a}} \approx 30$Billion years.
(2) What is before the big bang?
There are theories on this, but very difficult to check.
(3) What is the state/process right after the big bang is intensively studied research area.
(4) CMB is a remnant of big bang. LHC can be used to study some states after big bang , QGP.

11.7 Fate of the universe

The future evolution of the universe is determined by the Friedmann equations. From Eq (11.31)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{11.50}$$

using $H = \frac{\dot{a}}{a}$,

$$\frac{k}{a^2H^2} = \frac{8\pi\rho}{3H^2} - 1 \tag{11.51}$$

Defining critical density $\rho_{\text{crit}} = \frac{3H^2}{8\pi}$, and $\Omega = \frac{\rho}{\rho_{\text{crit}}}$, this becomes

$$\frac{k}{a^2H^2} = \Omega - 1 \tag{11.52}$$

Therefore,

$$\begin{aligned}
\rho < \rho_{\text{crit}} \Leftrightarrow \Omega < 1 \Leftrightarrow k = -1 \quad & \text{“open''} \\
\rho = \rho_{\text{crit}} \Leftrightarrow \Omega = 1 \Leftrightarrow k = 0 \quad & \text{“flat''} \\
\rho > \rho_{\text{crit}} \Leftrightarrow \Omega > 1 \Leftrightarrow k = +1 \quad & \text{“closed''}
\end{aligned}$$

• Note that Eq (11.52) should hold for all time t, including t_{now}. I.e., if we measure H, and then ρ, we can know our future.
• Observationally, we find $\rho \approx \rho_{\text{crit}}$ with very small error. Therefore the universe is flat.
• Among the entire energy density,
70%:Dark energy
25%:Dark matter

$$\text{5\%: Known matter}\begin{cases} 4\% & \text{interstellar Hydrogen and Helium} \\ 1\% & \begin{cases} 0.5\% & \text{stars} \\ 0.3\% & \text{neutrinos} \\ 0.03\% & \text{heavy elements, including most planet and human beings} \\ \ldots \end{cases} \end{cases}$$

11.8 Effects of G.R. in cosmology

11.8.1 Cosmological redshift

Consider how an comoving observer measure quantities of free falling objects, that is, quantities of geodesic motion.
• For an comoving observer,its four-velocity is

$$V^\mu = (1, 0, 0, 0)$$

Then we can form an tensor

$$K_{\mu\nu} \equiv a^2(g_{\mu\nu} + V_\mu V_\nu) \tag{11.53}$$

As one can verify, this satisfies

$$\nabla_{(\sigma} K_{\mu\nu)} = 0 \tag{11.54}$$

where $K_{\mu\nu}$ is a Killing tensor.
• Now suppose a particle moves along its geodesic. Its for-velocity is

$$V^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{11.55}$$

Then we can define

$$k^2 \equiv k_{\mu\nu} V^\mu V^\nu = a^2[V_\mu V^\nu + (V_\mu V^\mu)^2] \tag{11.56}$$

• Then I claim that k^2 is a constant of motion along the geodesics.

$$\begin{aligned}
\frac{d(k_{\mu\nu}V^\mu V^\nu)}{d\lambda} &= \frac{dx^\alpha}{d\lambda}\nabla_\alpha(K_{\mu\nu}V^\mu) \\
&= V^\alpha V^\mu V^\nu \nabla_\alpha k_{\mu\nu} + k_{\mu\nu}V^\alpha(V^\mu \nabla_\alpha V^\nu + V^\alpha \nabla_\alpha V^\mu) \\
&= 0 + 0
\end{aligned}$$

where Eq (11.54) and geodesic equation $V^\alpha \nabla_\alpha V^\mu = 0$ are used.
• Therefore for a timelike geodesic $V_\mu V^\mu = -1$,

$$\begin{cases} k^2 = a^2[-1 + (V^0)^2] \\ (V^0)^2 - g_{ij}V^i V^j = 1 \end{cases}$$

we solve $\vec{V} \equiv \sqrt{g_{ij}V^iV^j} = \frac{k}{a}$.
That is the particle will slow down w.r.t comoving observers/coordinates as a expands.
This is indeed a true, physically measurable slow down by comoving observers like us.
• For null geodesics $V_\mu V^\mu = 0$, we have $-V_\mu V^\mu = \frac{k}{a}$. Recall that $-V_\mu V^\mu$ is the energy of the light ray observed by the comoving observer, and $E = \hbar\omega$ for light rays, we have

$$\omega = -V_\mu V^\mu = \frac{k}{a} \tag{11.57}$$

At two different instents t_1 and t_2, then

$$\frac{\omega_1}{\omega_2} = \frac{a_2}{a_1} \tag{11.58}$$

The wavelength then follows

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1}{a_2} \tag{11.59}$$

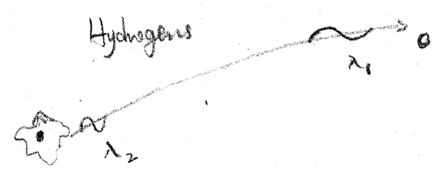
There will be a redshift as the universe expands, as observed by comoving observers. Usually, we measure the amount of resshift by

$$z \equiv \frac{\lambda_1 - \lambda_2}{\lambda_2} = \frac{\lambda_1}{\lambda_2} - 1 \tag{11.60}$$

where λ_1 is in the future of λ.

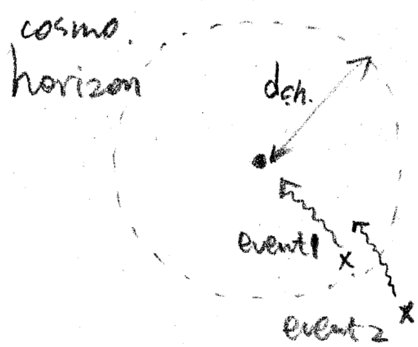
• z is a very useful quantity in observational cosmology. E.g., we can observe the hydrogen absorbtion line wavelength redshift. Say from $n = 2 \to n = 3$, $\lambda_2 = 656.3$nm, $\lambda_1 > \lambda_2$.

We can identify how much the universe has expanded from a particular event, such as supernova.



11.8.2 Cosmological horizon/particle horizon

• Since the universe has a finite age, and the light speed is the maximum speed that particles can travel and is also finite, there exist a boundary in space beyond which the light did not have enough time to travel to us (or any comoving observer), even if it started right after the big bang. This boundary is called cosmological horizon or particle horizon.



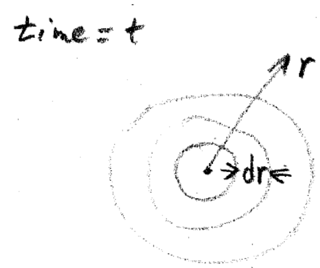
• Let us define a quantity to characterize the size of this horizon. We define a "proper-distance" to be the distance measured on a slice of constant time. At time t, the proper distance ds between two points with coordinate distance dr (same θ and φ), is defined through

$$ds^2 = 0 + a^2(t)\frac{dr^2}{1-kr^2} + 0 + 0$$

$$ds = \frac{a(t)}{\sqrt{1-kr^2}}dr$$

The horizon proper distance at time t is then

$$d_{c.h.} \equiv \int ds = a(t)\int_{r=0}^{r=r(horizon)} \frac{1}{\sqrt{1-kr^2}}dr \quad (11.61)$$



Now for the integral part, we know that light travels along null geodesics

$$ds^2 = 0 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + 0 + 0 \quad (11.62)$$

$$\frac{dt}{a(t)} = \frac{1}{\sqrt{1-kr^2}}dr \quad (11.63)$$

This produces

$$d_{c.h.}(t) = a(t)\int_{t'=0}^{t'=t} \frac{dt'}{a(t')} \quad (11.64)$$

• Therefore for different models of cosmology, dust, radiation etc, we can always compute a dc.h.
In particular, for any model we studied, we can check that

$$\lim_{t'\to 0} a(t') \to t'^{\alpha}, \quad 0 < \alpha < 1 \quad (11.65)$$

Therefore $d_{c.h}$ is finite.
• Observationally, $d_{c.h.}(now) \approx 13.7$ giga persec ≈ 45 Billion light years.
• There exist other kinds of horizons in some cosmological models. E.g., cosmological event horizon. Don't mix.

Chapter 12

Black Holes (B.H)

12.1 Schwarzschild B.H

12.2 Causal structure at $r > 2M$

The S.M. is given by

$$ds^2 = -\Big(1 - \frac{2M}{r})dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2 \tag{12.1}$$

• We have shown that the $r = 2M$ surface is not singular: curvatures are finite, and we also found explicit coordinate transforms.
• However, the $r = 2M$ still turns out to be an very interesting surface.
Consider a radial null geodesic, satisfying

$$ds^2 = 0 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2 \tag{12.2}$$

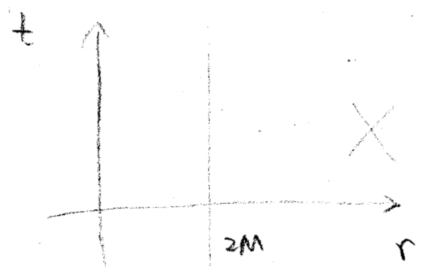
That is,

$$\frac{dt}{dr} = \pm\Big(1 - \frac{2M}{r}\Big)^{-1} \tag{12.3}$$

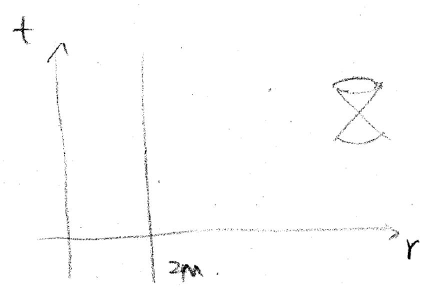
or,

$$\frac{dr}{dt} = \pm\Big(1 - \frac{2M}{r}\Big) \tag{12.4}$$

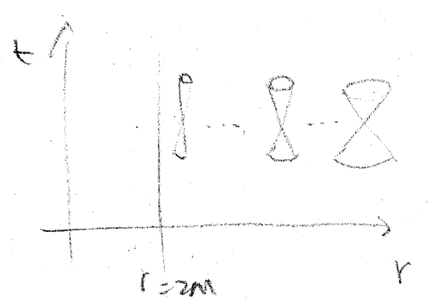
The "$-$" sign one is called in-going, while "$+$" is called out-going. Eq (12.4) gives the boundary of light cone.
At large r, $|\frac{dr}{dt}| \approx 1$.



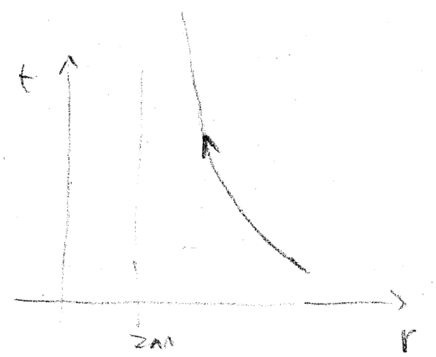
Since timelike particles travels slower, that is $|\frac{dr}{dt}| < |\frac{dr}{dt}|_{null}|$, the cone close like



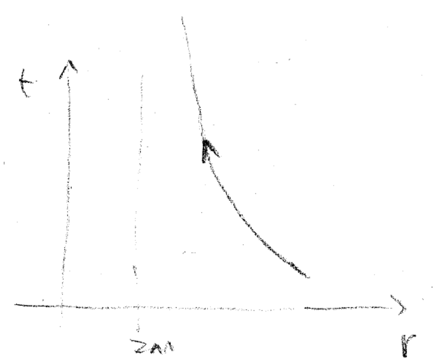
With smaller r, the cone closes up, until $r = 2M^{+}$, where the cone completely closes up.



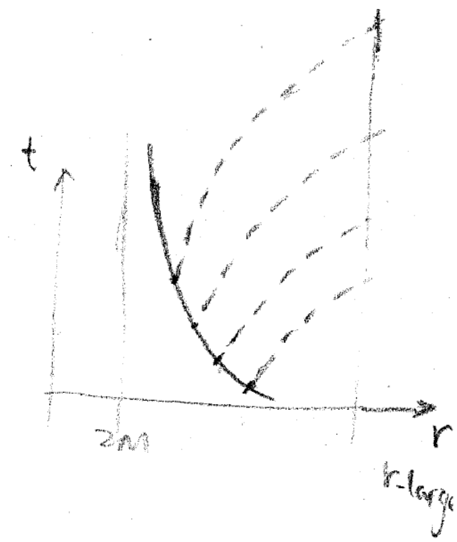
• It might seem that it will take infinite long t for the light ray to reach $2M$, and the light ray might never cross $r = 2M$. But indeed, infinite t does not mean infinite proper time for the traveler. t (or Δt) is only a good measurement of the proper time $r = \infty$ where metric is Minkovski.



• Infinite t to reach $r = 2M$ only means in the point of view of an asymptotic observer at $r = \infty$, the light ray/particle will never reach $r = 2M$ in finite time.
• If the traveler sends back a light signal at its own fixed frequency, from our studying of the red-shift, we know that the frequency at larger r is indeed smaller



$$\omega_{r-large} = \Big(\frac{1 - 2M/r_{small}}{1 - 2M/r_{large}}\Big)^{\frac{1}{2}} \omega_{r-small} \tag{12.5}$$



I.e., when $\omega_{r-small}$ and r_{large} are fixed, $\omega_{r-large}$ approaches infinity as r approaches $2M$.
• Indeed, for the proper time of the traveler (or affine parameter of the light ray), it only take finite amount of time to reach and then cross the $r = 2M$ surface. Too see this, we need to find a better coordinate system to describe the spacetime around $r = 2M$.

12.3 Coordinate transforms and causal structure around $r \lesssim 2M$

We first notice that the Eq (12.4)

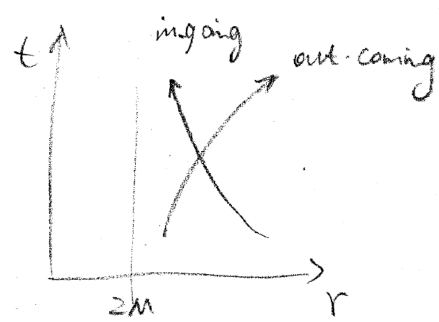
$$\frac{dt}{dr} = \pm\Big(1 - \frac{2M}{r}\Big)^{-1} \tag{12.6}$$

is solvable to get

$$t = \pm r^* + constant \tag{12.7}$$

$$r^*(r) = \int \frac{1}{1 - 2M/r} dr = r + 2M \ln |\frac{r}{2M} - 1| \tag{12.8}$$

I.e., the out-coming light ray move along curve $t - r^* =$ constant, the in-going light ray move along curve $t + r^* =$ constant



• This motivate us to try new coordinates defined by

$$u = t + r^* \tag{12.9}$$
$$v = t - R^* \tag{12.10}$$

where r^* is a function of r.
It is easy to work out relation from $(dt, dr) \to (du, dv)$.
The Eq (8.75) becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big) du dv + r^2 d\Omega^2 \tag{12.11}$$

$$\begin{cases} du = dt + \frac{1}{1-2M/r} dr \\ dv = dt - \frac{1}{1-2M/r} dr \end{cases}$$

where r should be thought as a function of u, v, obtainable by inversing $r^*(r) = \frac{1}{2}(u - v)$, using Eq (12.8)
The null geodesics are $u =$ constant or $v =$ constant.
This metric is manifestly non-singular at $r = 2M$. However, the $r = 2M$ surface, from Eq (12.8), Eq (12.9) and Eq () is at $u - v = -\infty$.
• We can try to make a half transform too:

$$(t, r) \to (u, r)$$

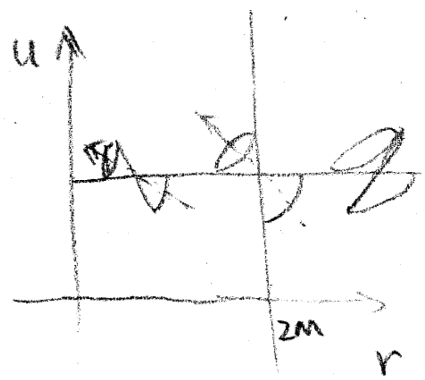
where u is given by Eq (12.9).
The metric Eq (8.75) simply becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big) du^2 + (dudr + drdu) + r^2 d\Omega^2 \tag{12.12}$$

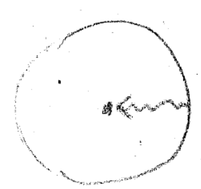
There is no singularity at $r = 2M$, which is also finite of the coordinate. This coordinate system is called Eddington-Finkelstein coordinates. The null geodesics along radial direction can be solved

$$\begin{aligned} & ds^2 = 0 \\ \Rightarrow & \frac{du}{dr} = \begin{cases} 0 & \text{“ingoing”} \\ 2\Big(1 - \frac{2M}{r}\Big)^{-1} & \text{“outcoming”} \end{cases} \end{aligned} \tag{12.13}$$

(1) Now its clear that when $r < 2M$, all future-directed null geodesics, $du > 0$, will have $dr < 0$, a decreasing r.
(2) No problem for both null and timelike geodesics to pass the $r = 2M$ surface. Therefore the $r = 2M$ surface is locally perfectly regular, globally function as a surface of no return.



Once a particle pass the $r = 2M$ surface, it can never come out.
The $r = 2M$ is given the name "event horizon" – any event happens inside that horizon can never be observed by an outsider in any long time.
The region bounded by $r = 2M$ is then called a "Black hole".



• Indeed from the definition of proper time.

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{12.14}$$

the maximum amount of proper time it takes for a timelike geodesic to travel from $r = 2M$ to $r = 0$ is when $dt^2 = 0 = d\Omega^2$.

$$\Delta\tau_{max} = \int d\tau = \int_{2M}^{0} -\frac{1}{\sqrt{\frac{2M}{r} - 1}} dr = \pi M \tag{12.15}$$

This is the proper time that you will take when you energy is 0^+ right at the surface $r = 2M$ after you enter the horizon.

12.4 Kuskal coordinates

• We can try to scale the $u - v = -\infty$ surface ($r = 2M$ surface) to somewhere finite. To do this, study the behavior of the metric near $r = 2M$.
From Eq (12.8),

$$r^* \simeq 2M \ln |r/2M - 1| \tag{12.16}$$
$$\Rightarrow r/2M \simeq 1 \pm e^{r^*/2M} = 1 \pm e^{(u-v)/4M} \tag{12.17}$$
$$\Rightarrow 1 - \frac{2M}{r} \simeq \pm e^{(u-v)/4M} \tag{12.18}$$

where "+" sign corresponds to $r > 2M$; "−" sign corresponds to $r < 2M$.
Then the metric Eq (12.11) around $r = 2M$ becomes

$$ds^2 = \mp (e^{u/4M} du)(e^{-u/4M} dv) + r^2 d\Omega^2 \begin{cases} \text{"}-\text{"} & r \approx 2M^+ \\ \text{"}+\text{"} & r \approx 2M^- \end{cases}$$

This motivates us to introduce another transform

$$V = \mp e^{-v/4M}, U = e^{u/4M} \begin{cases} \text{"}-\text{"} & r > 2M^+ \\ \text{"}+\text{"} & r < 2M^- \end{cases}$$

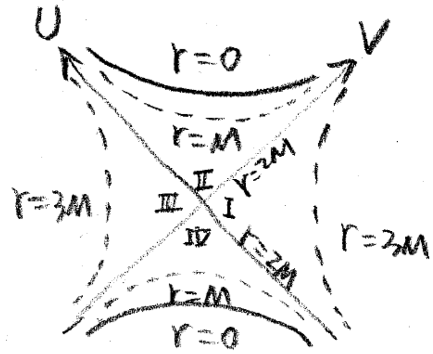
Then,

$$dV = -\frac{1}{4M} V dv, \quad dU = \frac{1}{4M} U du \tag{12.19}$$
$$UV = \mp e^{(u-v)/4M} = \mp e^{r^*/2M} = -e^{\frac{r}{2M}} (\frac{r}{2M} - 1) \tag{12.20}$$

The metric Eq (12.11) for all r becomes

$$\begin{aligned} ds^2 &= -\Big(1 - \frac{2M}{r}\Big)\Big(\frac{4M}{U}\Big)\Big(-\frac{4M}{V}\Big) dU dV + r^2 d\Omega \\ &= -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega \end{aligned}$$

The coordinates U and V are called null Kruskal coordinates.
• Kruskal diagram: A diagram obtained by drawing constant r and t lines on a U, V grid. We tilt the grid s.t. null geodesics are 45°



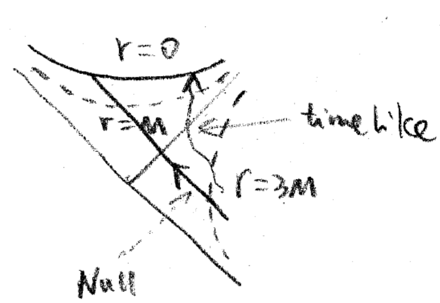
(1) From Eq (12.20), constant r curves are constant UV curves.
(2) $r = 2M$ are $U = 0$, $V = 0$ axis.
(3) Now there are two copies for each r = constant. Therefore the Kruskal coordinates reveals a large portion of manifold than covered by the original Schwarzschild coordinates.
(4) The original Schwarzschild coordinate cones region I and II. It is seen the hiting to $r = 0$ after $r < 2M$ is unavoidable.
• One can also do a rotation

$$\tilde{V} = (U + V)/2 \tag{12.21}$$
$$\tilde{U} = (U - V)/2 \tag{12.22}$$

The metric is then

$$ds^2 = \frac{32M^2}{r} e^{-r/2M} (d\tilde{U}^2 - d\tilde{V}^2) + r^2 d\Omega^2 \tag{12.23}$$

The $\tilde{U}$, $\tilde{V}$ are called Kruskal-Szekeres coordinates. $r = r(\tilde{U}, \tilde{V})$.



In this system, the relation (t, r) and $(\tilde{U}, \tilde{V})$ are worked out:

$$\left.\begin{array}{l}\tilde{U}\\ \tilde{V}\end{array}\right\} = \Big(\frac{r}{2M} - 1\Big)^{\frac{1}{2}} e^{\frac{r}{4M}} \left\{\begin{array}{l}\cosh(\dfrac{t}{4M})\\ \\ \sinh(\dfrac{t}{4M})\end{array}\right. \qquad r > 2M$$

$$\left.\begin{array}{l}\tilde{U}\\ \tilde{V}\end{array}\right\} = \Big(1 - \frac{r}{2M}\Big)^{\frac{1}{2}} e^{\frac{r}{4M}} \left\{\begin{array}{l}\cosh(\dfrac{t}{4M})\\ \\ \sinh(\dfrac{t}{4M})\end{array}\right. \qquad r < 2M$$

12.5 Reissner-Nordstrom BH

12.5.1 The metric

• Consider a spherecically symmetric metric

$$ds^2 = -e^{2\psi(t,r)} f dt^2 + f^{-1} dr^2 + r^2 d\Omega \tag{12.24}$$

where $f = f(t, r)$, $\psi = \psi(t, r)$. This is the most general form of a spherically symmetric spacetime.
Consider a electro-magnetic field in this spacetime. Its field strength $F^{\alpha\beta}$ has no θ or φ direction components. This ensures it's purely electric when measured by stationary observers.

$$[F^{\alpha\beta}] = \begin{bmatrix} 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12.25}$$

The Maxwell equation in vacuum

$$\nabla_\beta F^{\alpha\beta} = 0 = |g|^{\frac{1}{2}} \partial_\beta (|g|^{\frac{1}{2}} F^{\alpha\beta}) \tag{12.26}$$

That is,

$$\partial_r (e^\psi r^2 F^{tr}) = 0, \quad \partial_t (e^\psi r^2 F^{rt}) = 0 \tag{12.27}$$

solution then

$$F^{tr} = e^{-\psi} \frac{Q}{r^2}, \quad Q \text{ being an integral constant} \tag{12.28}$$

The energy momentum tensor

$$T^\alpha_\beta \equiv \frac{1}{4\pi} (F^{\alpha\mu} F_{\beta\mu} - \frac{1}{4} \delta^\alpha_\beta F^{\mu\nu} F_{\mu\nu}) \tag{12.29}$$

becomes

$$[T^\alpha_\beta] = \frac{Q}{4\pi r^4} \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{12.30}$$

• The E.E from Eq (8.1) becomes

$$\frac{\partial m(r,t)}{\partial r} = 4\pi r^2(-T^t_t) \tag{12.31}$$
$$\frac{\partial m(r,t)}{\partial t} = -4\pi r^2(-T^r_t) \tag{12.32}$$
$$\frac{\partial \psi(r,t)}{\partial r} = 4\pi r f^{-1}(-T^t_t + T^r_t) \tag{12.33}$$

where $f(r,t) = 1 - \dfrac{2m(r,t)}{r}$.
Substituting T^α_β

$$\begin{cases} \frac{\partial m}{\partial r} = \frac{Q^2}{2r^2} \\ \frac{\partial m}{\partial r} = 0 \end{cases}$$

we get $m(r) = M - \frac{Q^2}{2r}$, M being the integration constant.
• The metric becomes

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1} dr^2 + r^2 d\Omega^2 \tag{12.34}$$

One can show that Q is the total electric charge in the spacetime (we will not do this though).

12.5.2 B.H.

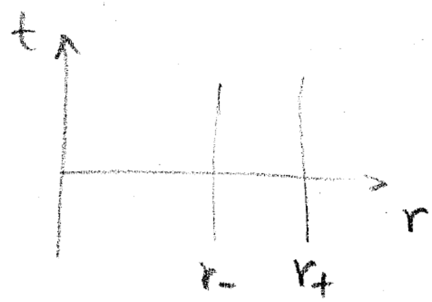
The metric components have coordinate singularities at

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 \tag{12.35}$$

or

$$\begin{cases} r_\pm = M \pm \sqrt{M^2 - Q^2} & |M| \geq |Q| \\ r = 0 & |M| < |Q| \end{cases}$$

• There are two surfaces, $r = r_+$ and $r = r_-$, $r_+ > r_-$, when $|M| > |Q|$. When $|M| = |Q|$, $r_+ \Leftrightarrow r_-$.
We can show that the r_+ is an event horizon and therefore metric Eq (12.34) contains a B.H, called Reissner-Nordstrom BH. When $Q = M$, this BH is called an extreme R.N B.H.



• The $r = r_-$ surface is not an event horizon, but an apparent horizon. (Which we have not teach the proper background to understand this.)
• The $r = 0$ point is a time-like singularity.
That is, travelers do not have to hit this point. They are free to avoid encounter it if they choose so.

12.6 Kerr B.H. and Kerr-Newnan B.H.

For completeness, we also give the last two of the four most well known BHs.

12.6.1 Kerr B.H.

The first is called a Kerr B.H. (discovered by Roy Kerr in 1963).
Its metric is

$$ds^2 = -\Big(1 - \frac{2Mr}{\rho^2}\Big)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \Sigma_{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{12.36}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{12.37}$$

$$\Delta = r^2 - 2Mr + a^2 \tag{12.38}$$

$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \tag{12.39}$$

This metric describes an stationary and mixing symmetric spacetime. M is the total mass and $L = aM$ is the angular momentum. So a us the ratio of angular momentum and mass. The metric describe the spacetime of an rotating body.
• One can show that it also has an event horizon when $M \geq a$. This is obtained from g_{rr} component, when $\Delta = 0$, we get

$$r = r_\pm = M \pm \sqrt{M^2 - a^2} \tag{12.40}$$

Therefore, Eq (12.36) also describes a block hole when $M \geq a$.

12.6.2 Kerr-Newman B.H.

The Kerr-Newman metric

$$ds^2 = -\Big(\frac{dr^2}{\Delta}+d\theta^2)\rho^2+(dt-a\sin^2\theta d\phi)^2\frac{\Delta}{\rho^2}-((r^2+a^2)d\phi-adt)^2\frac{\sin^2\theta}{\rho^2} \quad (12.41)$$

where

$$a = \frac{J}{M} \quad (12.42)$$

$$\rho^2 = r^2 + a^2\cos^2\theta \quad (12.43)$$

$$\Delta = r^2 - 2Mr + a^2 + \theta^2 \quad (12.44)$$

Here M is the total mass, J is angular momentum while Q is charge. When $M^2 > a^2 + Q^2$, the $\Delta = 0$ gives coordinate singularities

$$r = r_\pm = M \pm \sqrt{M^2 - (a^2 + \mathcal{O}^2)} \quad (12.45)$$

One can show that $r = r_+$ is an event horizon. Therefore the Eq (12.41) also can describe a B.H. called Kerr-Newman B.H.